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ABSTRACT

A variational principle governing the acoustic pressure on the exterior of an arbitrary body is derived from the Kirchhoff-Helmholtz integral relation. The principle is valid for acoustic radiation and diffraction problems. The general principle is specialized to the case of a thin body and then illustrated by an example of sound radiation from a flat, rigid, circular disk in transverse oscillation. The variational formulation has the surface pressure as the unknown variable, with the velocity normal to the surface taken as given. The Rayleigh-Ritz method is used to determine a solution in terms of truncated expansions of basis functions. The basis functions employed are polynomial and trigonometric functions, and piece-wise linear functions leading to a finite element description. The results compare very well with previous analytical investigations.

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A VARIATIONAL PRINCIPLE FOR SOUND RADIATION FROM VIBRATING BODIES

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1. INTRODUCTION

The prediction of sound radiation from vibrating bodies is a recurrent problem of practical importance in engineering science. Analytical solutions for such problems are generally limited to cases for which the surface of the object conforms to a suitable coordinate system such that the wave equation can be separated. For objects with nonstandard shapes, an approach commonly used is to reformulate the problem as a boundary integral relation, known as the Kirchhoff-Helmholtz integral theorem [Pierce, 1981], which enables one to express the acoustic field at any point exterior to the object as a definite integral over the surface of the object. Terms in the integrands involve both the pressure and the normal acceleration of the surface. These two surface quantities, however, are not independent and cannot be prescribed simultaneously. In most acoustic problems, one of the two is specified and it is necessary to solve the integral equation for the other unknown surface quantity.

Many numerical methods have been developed in that regard, for example, by Chen and Schweikert [1963], Banaugh and Goldsmith [1963], Chertock [1964], Copley [1967, 1968], Schenck [1968], Burton and Miller [1971], Bell et al [1978, 1979] for acoustic radiation and scattering problems. The present report describes a variational formulation of the fluid-structure interaction, which is derived from the Kirchhoff-Helmholtz integral theorem. The variational principle is attractive in that it always selects, from among various adjustable parameters of a chosen set of approximating functions, the ones that are optimal. It retains the accuracy of the boundary integral approaches, while offering the computational efficiency of approximate decoupling techniques, such as doubly asymptotic approximation developed by Geers [1971].

Although the formulation of an integral equation as a variational principle dated back as early as in 1884 [Volterra], it was not used for practical calculations in wave diffraction or scattering until the late 1940's. Levine and Schwinger [1948, 1949] employed a variational principle to study the diffraction of a scalar plane wave by an aperture in an infinite plane screen. Levine [1950], Bouwkamp [1954], and Sleator [1960] further discussed variational principles for acoustic diffraction and scattering problems.

In general, derivation of a useful variational principle from an integral equation requires that the integration kernel be symmetric, and therefore self-adjoint. The variational principle derived by Morse and Feshbach [1953] was obtained by simply taking a normal derivative of the Kirchhoff-Helmholtz integral relation for the pressure at an exterior point, and then letting the exterior point approach the surface. The resulting principle features an integration kernel that is self-adjoint. However, one of the resulting integrands becomes highly singular, and the integral is ambiguous without careful definition of how the singular integral is to be evaluated. It is not apparent how their principle could be numerically implemented for cases of interest.

In order to circumvent this singularity problem, the present report utilizes a method previously employed by Maue [1949] and Stallybrass [1967]. By using some mathematical identities associated with the free-space Green's functions, the integrand with non-integrable singularity is recast into a form involving the tangential derivative of the surface pressure. The singularities contained in the resultant integrands are at most of Cauchy type [Churchill, 1960], and therefore well behaved in the limit as the external point approaches the surface.

After we derive the present variational principle for an arbitrary body, we shall specialize it to situations where the body is slender, such that wetted surfaces are proximate. The application of the present variational principle to a circular disk with infinitesimal thickness in rigid transverse vibration is demonstrated. The numerical results agree remarkably well with those obtained by Leitner [1949].

2. KIRCHHOFF-HELMHOLTZ INTEGRAL THEOREM

The variational principle discussed in the present report is based on the standard Kirchhoff-Helmholtz integral theorem [Pierce, 1981] for the complex pressure amplitude p at an external field point $\mathbf{x}$ due to monochromatic excitation on a closed surface S. It is an exact corollary of the Helmholtz wave equation and the Sommerfeld radiation condition that, for $\mathbf{x}$ external to S, one has

$$p(\vec{x}) = -\frac{1}{4\pi} \iint_S \vec{n}(\vec{\xi}) \cdot \nabla_\xi p(\vec{\xi})\, G(\vec{x}|\vec{\xi})\, dA_\xi + \frac{1}{4\pi} \iint_S p(\vec{\xi})\, \vec{n}(\vec{\xi}) \cdot \nabla_\xi G(\vec{x}|\vec{\xi})\, dA_\xi \tag{1}$$

where $\vec{\xi}$ is a source point on S; the operator ∇_ξ denotes the gradient at that location, and dA_ξ is an element of area surrounding the source point. The quantity $G(\vec{x}|\vec{\xi})$ is the free space Green's function for the pressure at the field point $\vec{x}$ associated with a point source at $\vec{\xi}$ oscillating as $e^{-i\omega t}$; that is,

$$G(\vec{x}|\vec{\xi}) = \frac{1}{R} e^{ikR} \tag{2}$$

where $k = \omega/c$ (with c denoting the ambient speed of sound), and R is the distance between the source and field points,

$$R = |\vec{x} - \vec{\xi}| \tag{3}$$

Our interest here shall be focused on radiation problems, such that the surface velocity is presumed a given quantity, while the acoustic pressure is to be calculated. Euler's equation of motion for a fluid (the fluid dynamic counterpart of Newton's second law), combined with continuity of the normal component of particle velocity on the surface leads to

$$\vec{n}(\vec{\xi}) \cdot \nabla_\xi p(\vec{\xi}) = i\omega\rho v_n(\vec{\xi}) \tag{4}$$

where $v_n(\vec{\xi})$ is the complex amplitude of the (outward) normal component of the surface velocity.

Equations (1) and (4) show that, if the pressure and normal velocity on the surface S are known, then the pressure at any external point not on S may

be obtained from a definite integral. Our concern in this study is the evaluation of the pressure on S corresponding to the velocity distribution $v_n(\vec{\xi})$. Although we shall derive a different interrelationship between these two physical quantities on the surface, it is instructive to first review the derivation of a better known relationship [Schenck, 1968] that has often been used in earlier numerical studies. This relation is formally obtained by letting $\vec{x}$ approach the surface S.

3. FIRST INTEGRAL EQUATION FOR SURFACE PRESSURE

In the situation where $\vec{x}$ lies on S, such that $\vec{x} = \vec{\zeta}$, both integrals in Eq. (1) have a singularity at $\vec{\xi} = \vec{\zeta}$ because R = 0. The singularity in $G(\vec{x}|\vec{\xi})$ is integrable, but evaluation of the integral containing $\nabla_{\xi} G(\vec{x}|\vec{\xi})$ requires careful consideration. In particular, one finds that the value of the integral may have different values depending on whether one regards $\vec{x}$ as having approached the surface from the exterior or from the interior.

The analysis here of the effect of the singularity is similar to that of Kellogg [1953], in which the external point $\vec{x}$ is brought to a location $\vec{\zeta}$ on the surface in a limiting operation. As shown in Fig. 1, point $\vec{\zeta}$ is defined to be concurrent with the normal that intersects $\vec{x}$, so

$$\vec{x} = \vec{\zeta} + \epsilon\, \vec{n}(\vec{\xi}) \tag{5}$$

where ϵ is the small perpendicular distance of $\vec{x}$ from the surface. The region S' is a circular segment of S, centered at $\vec{\zeta}$, with small radius σ. The remainder of the surface is denoted as S". The required integral is evaluated by taking the limit as $\epsilon \to 0$ with σ fixed, and then taking the limit as $\sigma \to 0$. The order in which the two limits are taken is important.

Given that ϵ and σ are both small compared with any characteristic dimensions of the surface S and given that S is smooth near the point $\vec{\zeta}$, it is appropriate to regard S' as having the shape of an elliptical bowl. The

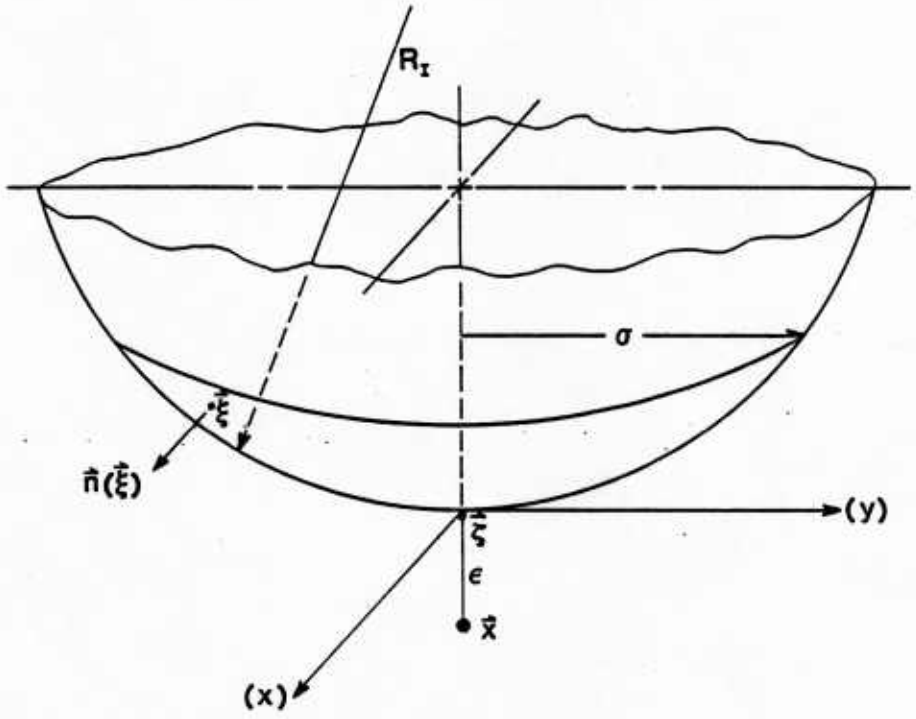

Figure 1. Integration when the field point approaches the surface.

principal curvature radii, R_I and R_{II}, of this bowl are those of the surface at location $\vec{\xi}$. In a local coordinate system with origin at $\vec{\xi}$, the surface S' is locally described by

$$z \simeq - x^2/(2R_I) - y^2/(2R_{II}) \tag{6}$$

(A convex surface would correspond to negative radii of curvature.) The unit outward normal vector $\hat{n}(\vec{\xi})$ at a generic point $\vec{\xi}$ on S' is given approximately by

$$\hat{n}(\vec{\xi}) \simeq \vec{e}_z + (x/R_I)\vec{e}_x + (y/R_{II})\vec{e}_y \tag{7}$$

In this local coordinate system, the external point $\vec{x}$ is at $\epsilon\vec{e}_z$, while the z-coordinate of $\vec{\xi}$ is as described by Eq. (6) above. Consequently, the vector $\vec{x}-\vec{\xi}$ is given by

$$R = \vec{x}-\vec{\xi} \simeq [\epsilon + (x^2/2R_I) + (y^2/2R_{II})]\vec{e}_z - x\vec{e}_x - y\vec{e}_y \tag{8}$$

and one derives

$$R\,\hat{n}(\vec{\xi})\cdot\nabla_{\xi}R \simeq -\epsilon + (x^2/2R_I) + (y^2/2R_{II}) \tag{9}$$

$$\hat{n}(\vec{\xi})\cdot\nabla_{\xi}G(\vec{\xi}|\vec{x}) \simeq (1/R^{-3} + k^2/2R)[\epsilon - (x^2/2R_I) - y^2/2R_{II}] \tag{10}$$

where, in the latter expression, the neglected terms are of the order of unity or smaller when ϵ and σ are both much smaller than R_I and R_{II}. Moreover, on the right side of Eq. (10), it is consistent to replace the distance R, wherever it appears, by

$$R \simeq (\epsilon^2 + r^2)^{1/2}; \qquad r^2 = x^2 + y^2 \tag{11}$$

and, for the differential of area, to set

$$dA_\xi \simeq r\,dr\,d\phi; \quad x = r\cos\phi; \quad y = r\sin\phi \tag{12}$$

where, for the surface S', the r-integration extends from 0 to σ, and the ϕ-integration extends from 0 to 2π.

Since all points on S" are at a finite distance from $\vec{x}$, the factor $\vec{n}(\vec{\xi})\cdot\nabla_\xi G(\vec{x}|\vec{\xi})$ has no singularities on S" when $\epsilon \to 0$. Moreover, if one lets $\epsilon \to 0$ first and considers r to be small but nonzero, then

$$\vec{n}(\vec{\xi})\cdot\nabla_\xi G(\vec{x}|\vec{\xi})\,dA_\xi \to -(1/2)(R_I^{-1}\cos^2\phi + R_{II}^{-1}\sin^2\phi)(1/r)\,r\,dr\,d\phi \tag{13}$$

The 1/r singularity here is cancelled by the factor in the area differential factor rdr. The integral in this limit is exactly the same as if one set $\vec{x}$ to $\vec{\zeta}$ at the outset, and then did the integral over the surface S. Hence, it is mathematically meaningful to integrate over the surface S" by taking the double limit such that first $\epsilon\to 0$, then $\sigma\to 0$. We therefore turn our attention to the contribution of S'.

Let $F(\vec{\xi})$ be any continuous scalar function on S. Then Eqs. (10) and (12) lead to

$$\begin{aligned}\iint_{S'} F(\vec{\xi})\ \vec{n}(\vec{\xi})\cdot\nabla_\xi G(\vec{x}|\vec{\xi})\ dA_\xi \\ \simeq \iint_{S'} F(\vec{\zeta})\ (\epsilon/R^3)\ r\,dr\,d\phi \\ + \iint_{S'} F(\vec{\zeta})\ \left[A(\epsilon,r,\phi) - (r/R)^2 B(\phi)\right] R^{-1} r\,dr\,d\phi\end{aligned} \tag{14}$$

where in the latter term we abbreviate

$$A(\epsilon,r,\phi) \simeq (1/2)k^2\epsilon - (1/4)k^2r^2(R_I^{-1}\cos^2\phi + R_{II}^{-1}\sin^2\phi) \tag{15a}$$

$$B(\phi) \simeq (1/2)(R_I^{-1}\cos^2\phi + R_{II}^{-1}\sin^2\phi) \tag{15b}$$

Functions A and B are both finite; A actually vanishes in the limit as both ϵ and σ go to zero. Also, we note that the quantity r/R is always less than 1, regardless of the value of ϵ. Consequently, the second term in Eq.(14) is of the order of magnitude of σ; it therefore vanishes in the limit $\sigma \to 0$.

We now proceed to evaluate the first term in Eq. (14). With ϵ small, but positive, an appropriate change of integration variable is to the polar angle θ, defined such that

$$r = \epsilon \tan\theta; \quad R = \epsilon \sec\theta; \quad dr = \epsilon \sec^2\theta \, d\theta; \quad (\epsilon/R^3)\, r\, dr = \sin\theta \, d\theta \tag{16}$$

The integration limits on θ are 0 and $\tan^{-1}(\sigma/\epsilon)$. In the limit as $\epsilon \to 0$ with σ fixed, this upper limit approaches $\pi/2$. The substitution of Eq. (16) into Eq. (14) makes the integration trivial, the overall result being simply $2\pi F(\vec{\zeta})$. Thus, we find that, as regards the integral over the entire surface, when ϵ goes from a finite positive value to 0,

$$\lim_{\epsilon \to 0} \iint_S F(\vec{\xi})\, \vec{n}(\vec{\xi}) \cdot \nabla_\xi G(\vec{x}|\vec{\xi})\, dA_\xi$$

$$= 2\pi\, F(\vec{\zeta}) + \iint_S F(\vec{\xi})\, \vec{n}(\vec{\xi}) \cdot \nabla_\xi G(\vec{\zeta}|\vec{\xi})\, dA_\xi \tag{17}$$

The result of applying Eq. (17) to the Kirchhoff-Helmholtz integral equation (1), with the pressure boundary condition given by Eq. (4), is

$$p(\vec{\zeta}) = -\frac{1}{2\pi}\iint_S i\omega\rho v_n(\vec{\xi})\, G(\vec{\zeta}|\vec{\xi})\, dA_\xi$$

$$+\frac{1}{2\pi}\iint_S p(\vec{\xi})\, \vec{n}(\vec{\xi})\cdot\nabla_\xi G(\vec{\zeta}|\vec{\xi})\, dA_\xi \qquad (18)$$

The above integral equation has been used often for numerical analysis of surface pressure, but the numerical implementations have not led to especially accurate results for many cases of practical interest. One of the intrinsic sources of difficulty [Rogers, 1973] is that, for certain discrete characteristic frequencies, the solution is not unique, because the homogeneous integral equation has eigensolutions (the multiplicative constant being arbitrary) at these frequencies. Another complication is that the integration kernel, $\vec{n}(\vec{\xi})\cdot\nabla_\xi G(\vec{x}|\vec{\xi})$, although integrable, is singular, so one is confronted with a singular integral equation. Also, the kernel is not symmetric in the interchange of $\vec{\xi}$ and $\vec{\zeta}$, so the linear operator associated with this integral equation is not self-adjoint.

4. SECOND INTEGRAL EQUATION FOR SURFACE PRESSURE

A strong case can be made that a variational principle is often a useful intermediate step for developing robust numerical methods for solving an integral equation. However, the integral equation (18) does not lead in a natural manner to a variational principle, because the integration kernel is not self-adjoint. Consequently, we have made use of a second integral equation (strictly speaking, a differential-integral equation) that applies to the same problem [Meyer et. al., 1978]. Although this equation is less well known, it nevertheless has the standard Kirchhoff-Helmholtz integral theorem as its foundation. We begin by using Eq.(1) to form the normal derivative of the pressure on the surface. Because of the singularity of the integral terms for a field point at the surface, we shall temporarily keep $\vec{x}$ off S by making use of Eq. (5). Thus, we form

$$\vec{n}(\vec{\zeta})\cdot\nabla_x p(\vec{x})$$

$$= -\frac{1}{4\pi}\iint_S i\omega\rho v_n(\vec{\xi})\ \vec{n}(\vec{\zeta})\cdot\nabla_x G(\vec{x}|\vec{\xi})\ dA_\xi$$

$$+\frac{1}{4\pi}\iint_S p(\vec{\xi})\ [\vec{n}(\vec{\zeta})\cdot\nabla_x]\ [\vec{n}(\vec{\xi})\cdot\nabla_\xi]\ G(\vec{x}|\vec{\xi})\ dA_\xi \qquad (19)$$

where ∇_x denotes the gradient at $\vec{x}$.

Note that the integrand in the second term of Eq. (19), in the limit as $\vec{x}\to\vec{\xi}$, is highly singular, so the appropriate expression for numerical evaluation of this term in the limit of $\epsilon \to 0$ has to be developed with some care. To this purpose, we follow an approach similar to that previously developed by Maue [1949] and Stallybrass [1967]. We first introduce the vector identity

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}) \qquad (20)$$

so that

$$(\vec{e}_i\times\nabla_x)\cdot(\vec{e}_j\times\nabla_\xi)G = (\vec{e}_i\cdot\vec{e}_j)(\nabla_x\cdot\nabla_\xi)G - (\vec{e}_j\cdot\nabla_x)(\vec{e}_i\cdot\nabla_\xi)G \qquad (21)$$

where $\vec{e}_i$ and $\vec{e}_j$ are unit vectors (possibly the same) in any cartesian coordinate system and G is the free-space Green's function (2). Since $G(\vec{x}|\vec{\xi})$ is a function only of $R = |\vec{x}-\vec{\xi}|$, $\nabla_\xi G(\vec{x}|\vec{\xi}) = -\nabla_x G(\vec{x}|\vec{\xi})$. This allows us to replace the third term on the right side of the above equation by an analogous expression, in which the subscripts i and j are interchanged. After making such a change, we multiply each term by $n_i(\vec{\zeta})n_j(\vec{\xi})$, and then sum over i and j from 1 to 3. A rearrangement of terms leads to

$$[\vec{n}(\vec{\zeta})\cdot\nabla_x]\ [\vec{n}(\vec{\xi})\cdot\nabla_\xi]G = [\vec{n}(\vec{\zeta})\cdot\vec{n}(\vec{\xi})]\ \nabla_x\cdot\nabla_\xi G$$

$$- [\vec{n}(\vec{\zeta})\ \times\ \nabla_x]\cdot[n(\vec{\xi})^{\rightarrow}\times\ \nabla_\xi]G \qquad (22)$$

Also, because $G(\vec{x}|\vec{\xi})$ has the property that $\nabla_\xi G(\vec{x}|\vec{\xi})$ is the negative of $\nabla_x G(\vec{x}|\vec{\xi})$, and because it satisfies the scalar Helmholtz equation, we can make the substitution

$$\nabla_x\cdot\nabla_\xi G(\vec{x}|\vec{\xi}) = -\nabla_x^2 G(\vec{x}|\vec{\xi}) = k^2 G(\vec{x}|\vec{\xi}) \qquad (23)$$

These two latter relations allow Eq. (19) to be recast in the form

$$\vec{n}(\vec{\zeta})\cdot\nabla_x p(\vec{x}) = -\frac{1}{4\pi}\iint_S i\omega\rho v_n(\vec{\xi})\ \vec{n}(\vec{\zeta})\cdot\nabla_x G(\vec{x}|\vec{\xi})\ dA_\xi$$

$$+\frac{k^2}{4\pi}\vec{n}(\vec{\zeta})\cdot\iint_S p(\vec{\xi})G(\vec{x}|\vec{\xi})\vec{n}(\vec{\xi})\ dA_\xi$$

$$-\frac{1}{4\pi}[\vec{n}(\vec{\zeta})\ \times\ \nabla_x]\cdot\iint_S p(\vec{\xi})[\vec{n}(\vec{\xi})\ \times\ \nabla\xi]\ G(\vec{x}|\vec{\xi})\ dA \qquad (24)$$

The second derivative of the Green's function will present a problem when $\epsilon\rightarrow 0$, so we integrate the last term by parts, such that

$$\iint_S p(\vec{\xi})[\vec{n}(\vec{\xi}) \times \nabla_\xi] G(\vec{x}|\vec{\xi}) \, dA_\xi$$

$$= \iint_S [\vec{n}(\vec{\xi}) \times \nabla_\xi] \{p(\vec{\xi}) G(\vec{x}|\vec{\xi})\} \, dA_\xi$$

$$- \iint_S [\vec{n}(\vec{\xi}) \times \nabla_\xi \vec{p}(\vec{\xi})] \, G(\vec{x}|\vec{\xi}) \, dA_\xi \qquad (25)$$

The first integral on the right side vanishes. In order to demonstrate this, we recall a version of Stokes' theorem [Less, 1950]. Let F be any scalar function that is continuous and that is differentiable with respect to tangential coordinates in S", and let C denote the curve bounding S". Then

$$\iint_{S"} \vec{n}(\vec{\xi}) \times \nabla_\xi F(\vec{\xi}) \, dA_\xi = \int_C F \, d\vec{\ell}_\xi \qquad (26)$$

where the line element $d\vec{\ell}$ progresses along C in the counter-clockwise sense when the unit normal $\vec{n}$ points toward the observer. In the present situation, F represents the factor contained within braces in the first integral in Eq. (25). Since $\epsilon \neq 0$, this factor F is finite. Also, since S is a closed surface, one can regard the open surface S" as being all of S except for a small patch surrounding any arbitrarily chosen point on S; in the limit as the dimensions of this patch shrink to zero, the curve C shrinks to zero length. It follows that the right side of Eq. (26) vanishes when S" becomes S. As a result, Eq. (24) becomes

$$\vec{n}(\vec{\zeta})\cdot\nabla_x p(\vec{x}) = -\frac{1}{4\pi}\, i\omega\rho \iint_S v_n(\vec{\xi})\ \vec{n}(\vec{\zeta})\cdot\nabla_x G(\vec{x}|\vec{\xi})\ dA_\xi$$

$$+\frac{k^2}{4\pi}\,\vec{n}(\vec{\zeta})\cdot\iint_S \vec{n}(\vec{\xi})\ G(\vec{x}|\vec{\xi})\ p(\vec{\xi})\ dA_\xi$$

$$+\frac{1}{4\pi}\,[\vec{n}(\vec{\zeta})\times\nabla_x]\cdot\iint_S G(\vec{x}|\vec{\xi})\ \vec{n}(\vec{\xi})\times\nabla_\xi p(\vec{\xi})\ dA_\xi \qquad (27)$$

The desired differential-integral equation emerges after one takes the limit of Eq. (27) as $\epsilon \to 0$; doing so yields

$$i\omega\rho U_n(\zeta) = \frac{k^2}{4\pi}\,\vec{n}(\vec{\zeta})\cdot\iint_S \vec{n}(\vec{\xi})\ G(\vec{\zeta}|\vec{\xi})\ p(\vec{\xi})\ dA_\xi$$

$$+\frac{1}{4\pi}\,[\vec{n}(\vec{\zeta})\times\nabla_\zeta]\cdot\iint_S G(\vec{\zeta}|\vec{\xi})\ \vec{n}(\vec{\xi})\times\nabla_\xi p(\vec{\xi})\ dA_\xi \qquad (28)$$

where we abbreviate

$$U_n(\vec{\zeta}) = v_n(\vec{\zeta}) + \lim_{\epsilon\to 0}\left[\frac{1}{4\pi}\iint_S v_n(\vec{\xi})\ \vec{n}(\vec{\zeta})\cdot\nabla_x G(\vec{x}|\vec{\xi})\ dA_\xi\right] \qquad (29a)$$

$$= \frac{1}{2}\,v_n(\vec{\zeta}) + \frac{1}{4\pi}\,PR\iint_S v_n(\vec{\xi})\ \vec{n}(\vec{\zeta})\cdot\frac{\vec{\zeta}-\vec{\xi}}{R}\,\frac{d}{dR}\left[\frac{e^{ikR}}{R}\right]dA_\xi \qquad (29b)$$

In the latter version, the symbol PR implies principal value and R is understood to be $|\vec{\zeta}-\vec{\xi}|$. The equivalence of these two versions of Eq. (29) can be demonstrated in a manner similar to that used in deriving Eq. (17). The

second version, however, has limited use; it should not be used, for example, if the vibrating body is an unbaffled plate (curved or not curved) with thickness idealized as being infinitesimal.

5. DERIVATION OF THE VARIATIONAL PRINCIPLE

We shall derive the variational principle from Eq. (28). The first step is to multiply each term of that equation by a virtual increment $\delta p(\vec{\zeta})$ in the actual pressure amplitude at the surface point $\vec{\zeta}$. Each term is then integrated over the surface S, with the parameter $\vec{\zeta}$ distinguishing the points on the surface during this integration. The result is then that

$$i\omega\rho \iint_S \delta p(\vec{\zeta})\, U_n(\vec{\zeta})\, dA_\zeta$$

$$= \frac{k^2}{4\pi} \iint_S \iint_S \delta p(\vec{\zeta})\, [\vec{n}(\vec{\zeta})\cdot\vec{n}(\vec{\xi})]\, p(\vec{\xi})\, G(\vec{\zeta}|\vec{\xi})\, dA_\xi dA_\zeta$$

$$+ \frac{1}{4\pi} \iint_S \delta p(\vec{\zeta})\, [\vec{n}(\vec{\zeta}) \times \nabla_\zeta] \cdot \left\{ \iint_S G(\vec{\zeta}|\vec{\xi})\, [\vec{n}(\vec{\xi}) \times \nabla_\xi] p(\vec{\xi})\, dA_\xi \right\} dA_\zeta \qquad (30)$$

An analysis (involving Stokes' theorem) similar to that described in Eqs. (24) and (25) applies to the last term in Eq. (30) with the inner integral (enclosed in large braces) regarded as a function $F(\vec{\zeta})$. Such a procedure enables one, in effect, to integrate by parts, transferring (with a change in sign) the operator $\vec{n}(\vec{\zeta}) \times \nabla_\zeta$ to the factor $\delta p(\vec{\zeta})$. The result is that

$$i\omega\rho \iint_S \delta p(\vec{\zeta})\, U_n(\vec{\zeta})\, dA_\zeta$$

$$= \frac{k^2}{4\pi} \iint_S \iint_S \delta p(\vec{\zeta})\, [\vec{n}(\vec{\zeta})\cdot\vec{n}(\vec{\xi})]\, p(\vec{\xi})\, G(\vec{\zeta}|\vec{\xi})\, dA_\xi dA_\zeta$$

$$- \frac{1}{4\pi} \iint_S \iint_S [\vec{n}(\vec{\zeta}) \times \nabla_\zeta \delta p(\vec{\zeta})]\cdot[\vec{n}(\vec{\xi}) \times \nabla_\xi p(\vec{\xi})] G(\vec{\zeta}|\vec{\xi})\, dA_\xi dA_\zeta \qquad (31)$$

We are seeking the pressure amplitude p at points on the surface S, given the normal velocity amplitude distribution $v_n(\xi)$. This means that $v_n(\xi)$ should be held constant when p is varied. Symmetry of the Green's function and the usual rules of the calculus of variables makes it possible to replace, within the integrands, variational factors of the generic form $F(\xi)\delta F(\zeta)$ by alternative variational factors $(1/2)\delta[F(\xi)F(\zeta)]$. Also, because the integration limits do not change during the variation, the integral of the variation is the variation of the integral. Similarly, a sum of variations can be replaced by the variation of the sum. All this allows us to recast Eq. (31) as the variational principle

$$\delta J[p] = 0 \qquad (32)$$

where the functional J[p] can be identified as

$$J[p] = -i\omega\rho \iint_S p(\vec{\zeta})\, 4\pi U_n(\vec{\zeta})\, dA_\zeta$$

$$+ \frac{k^2}{2} \iint_S \iint_S [\vec{n}(\vec{\zeta}) \cdot \vec{n}(\vec{\xi})]\, p(\vec{\zeta})\, p(\vec{\xi})\, G(\vec{\zeta}|\vec{\xi})\, dA_\xi dA_\zeta$$

$$- \frac{1}{2} \iint_S \iint_S [\vec{n}(\vec{\zeta}) \times \nabla_\zeta p(\vec{\zeta})] \cdot [\vec{n}(\vec{\xi}) \times \nabla_\xi p(\vec{\xi})] G(\vec{\zeta}|\vec{\xi})\, dA_\xi dA_\zeta \qquad (33)$$

The implication of Eq. (32) is that the functional J is stationary to small changes $\delta p(\xi)$ in the pressure distribution $p(\xi)$. One cannot, however, in general state that J has an extreme value (maximum or minimum) when the trial function $p(\xi)$ is equal to the true complex pressure distribution. If the trial function is taken to be the actual function plus $\epsilon f(\xi)$, where ϵ is small and $f(\xi)$ is a fixed admissible function, then J[p] can be regarded as a function of ϵ, and $dJ/d\epsilon$ must be zero at $\epsilon=0$. The sign of either the real or imaginary part of the second derivative $d^2J/d\epsilon^2$, however, depends on the choice of this fixed admissible funtion $f(\xi)$. In general, the class of admissible trial functions is restricted to functions that are continuous over the surface and piece-wise differentiable with respect to displacements on the surface, such that $\vec{n}(\xi) \times \nabla_\xi p(\xi)$ exists almost everywhere and is square integrable.

One possible trial function is simply a position-independent factor times the actual function. The stationarity condition is satisfied when this factor is unity, but differentiation with respect to this factor introduces additional factors of 1 and 2, respectively, for those terms in Eq. (33) which are linear and bilinear in the surface pressure. Consequently, after differentiating the functional with respect to the factor and then setting the factor equal to 1, one derives

$$i\omega\rho\iint_S p(\vec{\zeta})\ 4\pi U_n(\vec{\zeta})\ dA_\zeta =$$

$$+\ k^2\iint_S\iint_S [\vec{n}(\vec{\zeta})\cdot\vec{n}(\vec{\xi})]\ p(\vec{\zeta})\ p(\vec{\xi})\ G(\vec{\zeta}|\vec{\xi})\ dA_\xi dA_\zeta$$

$$-\iint_S\iint_S [\vec{n}(\vec{\zeta})\times\nabla_\zeta p(\vec{\zeta})]\cdot[\vec{n}(\vec{\xi})\times\nabla_\xi p(\vec{\xi})]G(\vec{\zeta}|\vec{\xi})\ dA_\xi dA_\zeta \qquad (34)$$

where the inserted function $p(\xi)$ is the actual complex pressure on the surface. (Note that the foregoing is merely the integral over dA_ζ of Eq. (28).) Substitution of Eq. (34) into Eq. (33) reveals that the stationary value of the functional J is numerically equal to

$$J[p_{true}] = -\ \frac{1}{2}\ i\omega\rho\iint_S p_{true}(\vec{\zeta})\ 4\pi U_n(\vec{\zeta})\ dA_\zeta \qquad (35)$$

Consequently, if a trial function for $p(\xi)$, that is correct to first order, is inserted into the functional J[p] stated in Eq. (33), one obtains an estimate, accurate to second order, of the surface integral of $p_{true}(\xi)U_n(\xi)$. In some cases, the latter may be of principal interest and may have an important physical identification; an accurate method for estimating its value would then be of great use.

6. INTEGRALS INVOLVING SURFACE VELOCITY

A few guidelines may be stated at the outset concerning the improper integral that causes the distinction between U_n and v_n appearing in Eqs. (29). We rewrite those relations as

$$U_n(\vec{\zeta}) - v_n(\vec{\zeta}) = \lim_{\epsilon \to 0}\left[\frac{1}{4\pi} \iint_S v_n(\vec{\xi})\ \vec{n}(\vec{\zeta}) \cdot \nabla_x G(\vec{x}|\vec{\xi})\ dA_\xi \right] \tag{36a}$$

$$= -\frac{1}{2} v_n(\vec{\zeta}) + \frac{1}{4\pi} PR \iint_S v_n(\vec{\xi})\ \vec{n}(\vec{\zeta}) \cdot \frac{\vec{\zeta}-\vec{\xi}}{R} \frac{d}{dR}\left[\frac{e^{ikR}}{R} \right] dA_\xi \tag{36b}$$

For thin platelike structures that are unbaffled, the surface S consists of two contiguous sheets with infinitesimal separation between them. If these sheets are labelled by the subscripts I and II, then adjacent points on the front and back of the surface can be labelled $\vec{\xi}_I$ and $\vec{\xi}_{II}$. The geometry requires

$$\vec{n}(\vec{\xi}_I) = -\vec{n}(\vec{\xi}_{II}) \tag{37}$$

so, if the surfaces are vibrating together as a unit, one must have

$$v_n(\xi_I) = -v_n(\xi_{II}) \tag{38}$$

Insofar as the integral on the right side of Eq. (36a) is concerned, the separation between $\vec{\xi}_I$ and $\vec{\xi}_{II}$ is much less than ϵ while the limit is being taken. Consequently R_I and R_{II} are the same for all finite ϵ; the opposite signs in Eq.(38) therefore require the integrations over the two sides of S to be equal but opposite for all finite ϵ; these integrals cancel and one is left with the simple result

$$U_n = v_n \tag{39}$$

for thin unbaffled plate-like bodies (alternately referred to as laminas).

For less specialized circumstances, a fail-safe method for evaluating the integral in Eq. (36a), without the complications of taking a limit or of separately distinguishing circumstances when portions of the body are laminas, can be developed from the identity

$$\lim_{\epsilon \to 0} \iint_S \vec{n}(\vec{\xi}) \cdot \nabla_x (R^{-1}) \, dA_\xi = 0 \tag{40}$$

which follows from (i) the fact that ∇_x applied to 1/R is equivalent to $-\nabla_\xi$ applied to 1/R, (ii) Gauss's theorem, and (iii) the fact that 1/R satisfies Laplace's equation. Consequently, one can write

$$4\pi\left[U_n(\vec{\zeta}) - v_n(\vec{\zeta})\right] = \lim_{\epsilon \to 0} \iint_S v_n(\vec{\xi}) \, \vec{n}(\vec{\zeta}) \cdot \nabla_x G(\vec{x} | \vec{\xi}) \, dA_\xi$$

$$= \iint_S \left[v_n(\vec{\xi}) \vec{n}(\vec{\zeta}) - v_n(\vec{\zeta}) \vec{n}(\vec{\xi})\right] \cdot \frac{\vec{\zeta} - \vec{\xi}}{R} \frac{d}{dR}\left(\frac{e^{ikR}}{R} \right) dA_\xi$$

$$+ \; v_n(\vec{\zeta}) \iint_S \vec{n}(\vec{\xi}) \cdot \frac{\vec{\zeta} - \vec{\xi}}{R} \frac{d}{dR}\left(\frac{e^{ikR}}{R} - \frac{1}{R} \right) dA_\xi \tag{41}$$

Here the singularities in the integrands of the integrals in the second version are at most of order 1/R (which is integrable), so one need not be concerned with explicitly taking the limit as $\epsilon \to 0$.

An example for which the above representation might be computationally useful is that of a finite length circular cylinder vibrating as a rigid body parallel to its axis. If the length L of the cylinder goes to zero, then one has a circular disk (discussed in greater depth in the next section) in

transverse vibration, and Eq. (39) should apply, $U_n = v_n$. This limit, however, does not emerge very easily from Eq. (36b), but it does pop out of Eq. (41). In the limit of L→0, the area integrals in (41) are over only the top (z = L/2) and bottom surfaces (z = - L/2) of the cylinder. If $\boldsymbol{\zeta}$ is a point on the top surface, then $\hat{\mathbf{n}}(\boldsymbol{\zeta})$ is in the direction of the unit vector $\hat{\mathbf{e}}_z$, while $\hat{\mathbf{n}}(\boldsymbol{\xi})$ may be in either the $\hat{\mathbf{e}}_z$ or $-\hat{\mathbf{e}}_z$ directions. With L→0, one has in any case that $\boldsymbol{\zeta}$-$\boldsymbol{\xi}$ is perpendicular to both $\hat{\mathbf{n}}(\boldsymbol{\xi})$ and $\hat{\mathbf{n}}(\boldsymbol{\zeta})$, so both the first and second terms in Eq. (41) vanish trivially in this limit. One cannot draw such a conclusion for the integral in Eq. (36b), because the principal value applies only for the integral over the top surface (given that $\boldsymbol{\zeta}$ is on the top surface). For the integral over the bottom end, one must in general evaluate that integral for finite L, then take the limit as L→0. One would get a different (and incorrect) answer if one jumped inside the integral and took the limit as L→0 before evaluating the limit. Such problems do not arise, however, for the integrals in the latter version of Eq. (41), because the integrands are sufficiently well-behaved and non-singular that the order of taking the limit and carrying out the integrations can be freely interchanged.

7. SPECIAL CASE OF A THIN DISK

Because there are no gradients of the Green's function within the integrands of the second and third terms of the functional J[p] in Eq. (33), one need not characterize those integrals as being Cauchy principal values. Consequently, the formalism is easily adapted to a slender body, for which one region of the surface S is infinitesimally close to a different region of S.

For example, consider the disk in Fig. 2, which is shown on edge. Let us denote variables on the upper and lower surfaces by a subscript plus or minus, respectively. The disk is assumed to be vibrating as a rigid body normal to its faces, so the analysis given in the preceding section applies and one has $U_n = v_n$. Also, as stated in Eq. (38), the surface velocities $v_n(\boldsymbol{\xi}_+)$ and $v_n(\boldsymbol{\xi}_-)$ are 180° out-of-phase, that is, when the upper surface moves inward, the lower surface moves outward, and vice versa. It follows that the actual

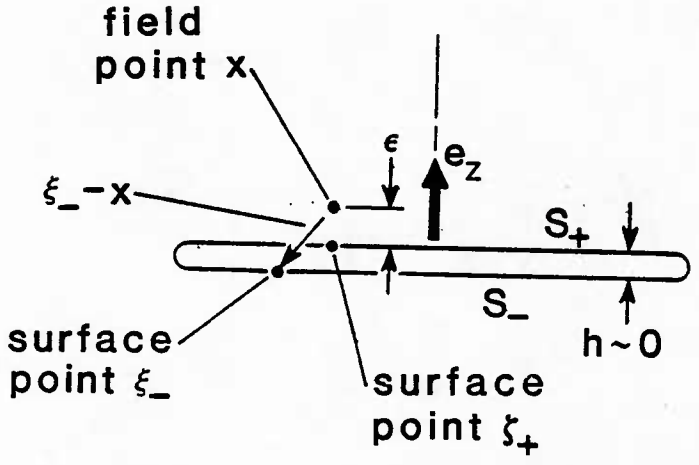

Figure 2. Field and surface points for a thin disk.

acoustic pressure must have comparable behavior, and must be antisymmetric with respect to reflection through the plane on which the disk nominally lies. We choose to restrict our class of admissible trial functions for p to include only functions that have this property. Thus we have

$$v_n(\vec{\xi}_-) = -v_n(\vec{\xi}_+), \qquad p(\vec{\xi}_-) = -p(\vec{\xi}_+) \tag{42}$$

$$v_n(\vec{\xi}_-)p_n(\vec{\xi}_-) = v_n(\vec{\xi}_+)p(\vec{\xi}_+) \tag{43}$$

Also, since the unit normal vectors on opposite sides of the disk are oppositely directed, we have

$$\vec{n}(\vec{\xi}_-)p(\vec{\xi}_-) = \vec{n}(\vec{\xi}_+)p(\vec{\xi}_+); \quad \vec{n}(\vec{\xi}_-) \times \nabla_{\xi} p(\vec{\xi}_-) = \vec{n}(\vec{\xi}_+) \times \nabla_{\xi} p(\vec{\xi}_+) \tag{44}$$

These relations, of course, also hold if the dummy integration variable $\vec{\xi}$ is replaced by the dummy integration variable $\vec{\zeta}$.

Equations (42-44) substantially simplify the functional J[p]. When each of the integrals in Eq. (33) is decomposed into the contributions of the upper and lower surfaces, we find that J[p] may be expressed in terms of integrals extending over the upper surface only; specifically,

$$J[p] = -8\pi i\omega\rho \iint_{S_+} v_n(\vec{\xi})p(\vec{\xi})\, dA_{\xi} + 2k^2 \iint_{S_+}\iint_{S_+} p(\vec{\zeta})p(\vec{\xi})G(\vec{\zeta}|\vec{\xi})\, dA_{\xi}dA_{\zeta}$$

$$- 2\iint_{S_+}\iint_{S_+} [\vec{e}_z \times \nabla_{\zeta} p(\vec{\zeta})] \cdot [\vec{e}_z \times \nabla_{\xi} p(\vec{\xi})] G(\vec{\zeta}|\vec{\xi})\, dA_{\xi}dA_{\zeta} \tag{45}$$

where $\vec{\zeta}$ and $\vec{\xi}$ are dimensionless variables and can

8. ANALYSIS OF A RIGID DISK WITH DISTRIBUTED BASIS FUNCTIONS

In this section we shall represent the pressure distribution on the surface of an unbaffled rigid disk in a series of assumed basis functions, which are each in general nonzero at every point on the disk. It is convenient to begin by nondimensionalizing the functional J in Eq. (45). For this purpose we scale distance by the radius a of the disk, and scale pressure by $\rho c v_o$, where ρ and c are the ambient density and sound speed, and v_o is the amplitude of the surface velocity, such that on the plus side of the disk, the normal velocity's complex amplitude is given by

$$v_n(\vec{\xi}) = v_o \tag{46}$$

Let an overcarat denote a nondimensional quantity. We then find that Eq. (45) becomes

$$\hat{J}[\hat{p}] = \frac{J}{4\rho^2 c^2 v_o^2 a} =$$

$$- 2\pi i k a \iint_{\hat{S}_+} \hat{p}(\hat{\xi})\, d\hat{A}_\xi + \frac{(ka)^2}{2} \iint_{\hat{S}_+} \iint_{\hat{S}_+} \hat{p}(\hat{\zeta})\hat{p}(\hat{\xi})\hat{G}(\hat{\zeta}|\hat{\xi})\, d\hat{A}_\xi d\hat{A}_\zeta$$

$$- \frac{1}{2} \iint_{\hat{S}_+} \iint_{\hat{S}_+} [\vec{e}_z \times \hat{\nabla}_\zeta \hat{p}(\hat{\zeta})] \cdot [\vec{e}_z \times \hat{\nabla}_\xi \hat{p}(\hat{\xi})] \hat{G}(\hat{\zeta}|\hat{\xi})\, d\hat{A}_\xi d\hat{A}_\zeta \tag{46}$$

where

$$\hat{p} = \frac{p}{\rho c v_o}; \qquad \hat{\nabla} = a\nabla; \qquad \hat{G} = a\,G = \frac{e^{ika\hat{R}}}{\hat{R}}; \qquad d\hat{A} = \frac{1}{a^2} dA \tag{47}$$

Because of the axial symmetry of the surface velocity, the pressure amplitude on the disk must only be a function of the radial distance from the center of the disk. Polar coordinates for the disk are depicted in Fig. 3, where ra and sa are defined as the radial distances to the points $\vec{\zeta}$ and $\vec{\xi}$, respectively. Thus r and s are dimensionless variables that may be regarded as the possible arguments (which replace $\vec{\zeta}$ and $\vec{\xi}$) of the scaled dimensionless pressure. Correspondingly, we find that

$$\hat{p}(\hat{\zeta}) = \hat{p}(r); \qquad \hat{p}(\hat{\xi}) = \hat{p}(s);$$

$$\hat{\nabla}_{\zeta}\hat{p}(\hat{\zeta}) = \hat{p}'(r)\vec{e}_r; \qquad \hat{\nabla}_{\xi}\hat{p}(\hat{\xi}) = \hat{p}'(s)\vec{e}_s;$$

$$\hat{G}(\hat{\zeta}|\hat{\xi}) = \frac{e^{ik\hat{a}\hat{R}}}{\hat{R}} \tag{48}$$

where a prime indicates a derivative with respect to the argument, and $\vec{e}_r$ and $\vec{e}_s$ are the radial unit vectors shown in Fig. 3. The nondimensional distance $\hat{R}$ is found from the law of cosines to depend on the relative polar angle θ (equal to $\theta_r - \theta_s$), according to the well known relation

$$\hat{R} = (r^2 + s^2 - 2rs\cos\theta)^{1/2} \tag{49}$$

The scaled elements of area in this axisymmetric situation may be taken as

$$d\hat{A}_{\zeta} = rd\theta_r dr \quad , \quad d\hat{A}_{\xi} = sd\theta_s ds \tag{50}$$

Since the integrand for the double integrations over area depends only on the relative angle θ, a simple transformation to θ as a variable of integration, replacing, say, θ_r, enables one of the angular integrations to be done trivially, yielding a factor of 2π. This leads to the following simple form for the scaled functional that appears in Eq. (46)

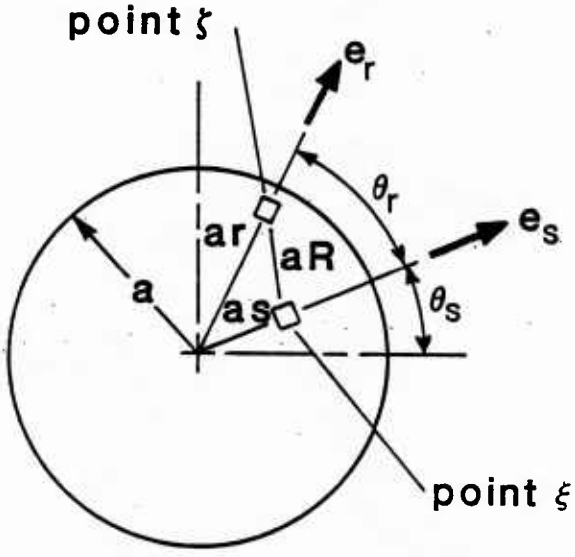

Figure 3. Position coordinates for dual integration.

$$\frac{\hat{J}}{2\pi} = -2\pi ika\int_0^1 \hat{p}(r)r\,dr + \frac{(ka)^2}{2}\int_0^1\int_0^1 \hat{p}(r)\hat{p}(s)C_1(r|s)\,drds$$

$$- \frac{1}{2}\int_0^1\int_0^1 \hat{p}'(r)\hat{p}'(s)C_2(r|s)\,drds \qquad (51)$$

where

$$C_1(r|s) = rs\int_0^{2\pi} \frac{e^{ika\hat{R}}}{\hat{R}}\,d\theta \qquad (51a)$$

$$C_2(r|s) = rs\int_0^{2\pi} \frac{e^{ika\hat{R}}}{\hat{R}}\cos\theta\,d\theta \qquad (52b)$$

Note that the factor 2π occurs in Eq. (51) because the integrands in Eq. (46) are dependent on the polar angles θ_s and θ_r only through their difference, which is here denoted as θ. The differential factors $C_1(r|s)$ ds dr and $C_2(r|s)$ ds dr can be interpreted as describing the effect of an annulus of radius s and width ds on another annulus of radius r and width dr. We shall refer to these functions C_1 and C_2 as the integrated Green's functions.

The basis functions selected to represent the pressure distribution on the surface of the disk must satisfy all requirements that are imposed by the manner of derivation of the variational principle on admissible trial functions. In particular, these must be continuous and differentiable, so that derivatives such as $\hat{p}'(r)$ exist. The derivation exploited the fact that the pressure at corresponding locations on the upper and lower surface differ only in sign. However, because the thickness of the disk is infinitesimal, the essure on the two surfaces at the edge must be equal. Both conditions can

only be satisfied if the pressure vanishes at the edge, so

$$\hat{p}(r) = 0 \qquad \text{at } r = 1 \tag{53}$$

Since the admissible trial functions must be continuous over the disk's surface, each of the basis functions must satisfy the above condition. Thus the general trial function formed as a linear combination of N basis functions can be written

$$\hat{p}(r) = \sum_{n=1}^{N} P_n \Psi_n(r); \qquad \Psi_n(1) = 0 \tag{54}$$

Note that the condition $p' = 0$ at $r = 0$, which results from axisymmetry, is a natural boundary condition that will emerge from the variational principle for the exact solution, but it is not a requirement that must be imposed at the outset on trial functions and on variations. It is not necessary that the basis functions $\Psi_n(r)$ satisfy this condition.

When we substitute Eq. (54) and a similar expression for $p(s)$ into Eq. (51), we find that $\hat{J}$ is a quadratic polynomial in the complex coefficients P_n:

$$\frac{\hat{J}}{2\pi} = \frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N} a_{jn}P_jP_n - \sum_{n=1}^{N} b_nP_n \tag{55}$$

$$a_{jn} = a_{nj} = (ka)^2\int_0^1\int_0^1 \Psi_j(r)\Psi_n(s)C_1(r|s)\,drds$$

$$-\int_0^1\int_0^1 \Psi_j'(r)\Psi_n'(s)C_2(r|s)\,drds \tag{56}$$

$$b_n = 2\pi ika\int_0^1 r\Psi_n(r)dr \tag{57}$$

The task of evaluating the coefficients a_{jn} and b_n shall be addressed in the next section. Once they are known, the variation of J[p] is obtained from virtual increments in each of the pressure coefficients. Those increments are arbitrary, so $\delta\hat{J} = 0$ for all admissible variations in the trial function (54) requires that

$$\frac{\partial\hat{J}}{\partial P_n} = 0 \quad \text{for } n = 1,2,\ldots,N \tag{58}$$

In view of the form of $\hat{J}$ indicated by Eq. (55), we conclude that the coefficients are governed by

$$[a]\{P\} = \{b\} \tag{59}$$

where the elements of the square array [a], and of the vectors {b} and {P} are the a_{mn}, b_n, and P_n, respectively. After the set of linear equations represented by Eq. (59) has been solved for the coefficients P_n, it is a simple matter to recreate the spatial distribution $\hat{p}(r)$ from Eq. (54).

9. EVALUATION OF THE INTEGRATED GREEN'S FUNCTIONS

A singularity in each of the integrated Green's functions, given by Eqs. (52), arises when the source radial coordinate s approaches the field radial coordinate r. In such a circumstance, the separation distance $\hat{R}$, which appears in the denominator of the respective integrands, vanishes for $\theta = 0$. Our approach to evaluating the $C_j(r|s)$ will separate out their singular parts as elliptic integrals.

The analysis and the writing of the arguments of the elliptic integrals are facilitated if one introduces the abbreviations

$$m = \frac{4rs}{(r+s)^2}; \qquad \alpha = ka(r+s) \tag{60}$$

so that Eqs. (52) become

$$C_1(r|s) = \frac{1}{2}\, m(r+s) \int_0^{\pi} \frac{e^{i\alpha D}}{D}\, d\theta \tag{61a}$$

$$C_2(r|s) = \frac{1}{2}\, m(r+s) \int_0^{\pi} \frac{e^{i\alpha D}}{D} \cos\theta\, d\theta \tag{61b}$$

where

$$D = \frac{\hat{R}}{r+s} = \left[1 - m \cos^2(\theta/2)\right]^{1/2} \tag{62}$$

The factor $\cos\theta$ in the integrand of the expression (61a) for $C_2(r|s)$ is eliminated with the aid of a trigonometric identity

$$\cos\theta = 2\cos^2(\theta/2) - 1 = \frac{2-m}{m} - \frac{2}{m}D^2 \tag{63}$$

As a result, the integrated Green's functions may be obtained from two basic integrals, G_1 and G_2, according to

$$C_1(r|s) = m(r+s)G_1(\alpha,m) \tag{64a}$$

$$C_2(r|s) = (r+s)\left[(2-m)G_1(\alpha,m) - 2G_2(\alpha,m)\right] \tag{64b}$$

where

$$G_1(\alpha,m) = \frac{1}{2}\int_0^{\pi} \frac{e^{i\alpha D}}{D}\, d\theta \tag{65a}$$

$$G_2(\alpha,m) = \frac{1}{2}\int_0^{\pi} D\, e^{i\alpha D}\, d\theta \tag{65b}$$

Both of the above expressions reduce to elliptic integrals when α is set to zero. To highlight the resemblance of the functions $G_j(\alpha,m)$ to elliptic integrals, Euler's identity and the trigonometric half-angle formulas are combined. The exponential factor then becomes

$$e^{i\alpha D} = \left[1 - 2\sin^2\left(\frac{\alpha D}{2}\right)\right] + 2i\sin\left(\frac{\alpha D}{2}\right)\cos\left(\frac{\alpha D}{2}\right) \tag{66a}$$

$$= 1 + 2i\sin\left(\frac{\alpha D}{2}\right)\exp\left(\frac{i\alpha D}{2}\right) \tag{66b}$$

We now introduce a change of variable for the polar angle θ,

$$\phi = \frac{\pi}{2} - \frac{\theta}{2} \tag{67}$$

so that the distance parameter D becomes

$$D = \left[1 - m \sin^2\phi\right]^{1/2} \tag{68}$$

Substitution of Eqs. (66)-(68) converts Eqs. (65) to

$$G_1(\alpha,m) = K(m) + 2i \int_0^{\pi/2} \frac{1}{D} \sin\left[\frac{\alpha D}{2}\right] \exp\left[\frac{i\alpha D}{2}\right] d\phi \tag{69a}$$

$$G_2(\alpha,m) = E(m) + 2i \int_0^{\pi/2} D \sin\left[\frac{\alpha D}{2}\right] \exp\left[\frac{i\alpha D}{2}\right] d\phi \tag{69b}$$

where K(m) and E(m) are the complete elliptic integrals [Milne-Thomson, 1972] of the first and second kind, respectively,

$$K(m) = \int_0^{\pi/2} \frac{d\phi}{(1-m \sin^2\phi)^{1/2}} \tag{70a}$$

$$E(m) = \int_0^{\pi/2} (1-m \sin^2\phi)^{1/2} d\phi \tag{70b}$$

The singularity associated with the field point approaching the source point corresponds now to m = 1 and $\phi = \pi/2$, in which case D = 0. Both integrands in Eqs. (69) have a finite limit as D→0, so their evaluation

involves a straightforward application of conventional numerical methods. We use Simpson's rule, based on subdividing the interval $0 \le \phi \le \pi/2$ sufficiently finely to resolve the oscillations in the sinusoidal functions forming the integrand. Corresponding values for the elliptic integrals were obtained from polynomial approximations [Milne-Thomson, 1972]. Such are conveniently expressed in terms of the complementary parameter

$$m_1 = 1 - m \tag{71}$$

Then, to an absolute error that does not exceed 2×10^{-8}, we have

$$K(m) = \sum_{j=0}^{4} (a_j - b_j \ln m_1) m_1^{\,j} \tag{72a}$$

$$E(m) = 1 + \sum_{j=1}^{4} (c_j - d_j \ln m_1) m_1^{\,j} \tag{72b}$$

where the numerical coefficients are

$$
\begin{aligned}
a_0 &= (1/2)\ln(16) & b_0 &= 0.5 \\
&= 1.38629436112 & & \\
a_1 &= 0.09666344259 & b_1 &= 0.12498593597 \\
a_2 &= 0.03590092383 & b_2 &= 0.06880248576 \\
a_3 &= 0.03742563713 & b_3 &= 0.03328355346 \\
a_4 &= 0.01451196212 & b_4 &= 0.00441787012 \\
c_1 &= 0.44325141463 & d_1 &= 0.24998368310 \\
c_2 &= 0.06260601220 & d_2 &= 0.09200180037 \\
c_3 &= 0.04757383546 & d_3 &= 0.04069697526 \\
c_4 &= 0.01736506451 & d_4 &= 0.00526449639
\end{aligned}
\tag{73}
$$

The logarithmic singularity in $K(m)$ as $m \to 1$ causes both of the integrated Green's functions $C_j(r|s)$ to become singular as $s \to r$. Although the manner of derivation of Eq. (56), which defines the matrix coefficient a_{jn}, indicates that the requisite integrals must exist, and that the singularities in the integrands must be integrable, it is instructive to demonstrate this afresh taking into account the explicit knowledge of the logarithmic singularities. We accordingly digress to give a brief proof that the contribution of the singularity of $K(m)$ to these integrals is finite.

When m is very close to unity, $m \simeq 1$, the elliptic integral $E(m)$ is also nearly unity (and therefore finite), but the elliptic integral of the first kind has the behavior

$$
K(m) \simeq \frac{1}{2} \ln\left(\frac{16}{1 - m}\right) \tag{74}
$$

For the integrations involved in the expression (56) for the matrix coefficient a_{jn}, the region in the integration plane surrounding the singularity can be taken to be the strip along the diagonal described by $r-\Delta \leq s \leq r+\Delta$, where Δ is very small. In this region, the dominant contributions to the $C_j(r|s)$

factors in the integrand are identified from Eqs. (64) and (69) to be

$$C_1(r|s) \simeq 2rmK(m); \quad C_2(r|s) \simeq 4rK(m)$$

If one assumes that the derivatives $\Psi'(r)$ of the basis functions are finite, then the corresponding contributions from such singular terms to the integrals in Eq. (56), which define the a_{jn}, can be generically written

$$\{\text{Contribution to } a_{jn}\} \simeq \int_0^1 \int_{r-\Delta}^{r+\Delta} F(r,s)\, r\, K(m)\, ds\, dr + O(\Delta) \tag{75}$$

where, with the assumption previously stated, $F(r,s)$ denotes a factor that is finite in the domain of integration.

Because of the smallness of Δ, the mean value theorem of the integral calculus allows us to neglect fluctuations in $F(r,s)$ resulting from changing s, so we replace $F(r,s)$ by $F(r,r)$. Then, in order to perform the integral over the domain of s, we change the variable of integration to ξ, such that

$$s=r(1+\xi), \qquad 1-m = \left(\frac{s-r}{s+r}\right)^2 \simeq \frac{1}{4}\,\xi^2; \qquad K(m) \simeq \frac{1}{2}\ln\left[64/\xi^2\right] \tag{76}$$

Such substitutions transform Eq. (75) to

$$\{\text{Contribution to } a_{jn}\} \simeq \int_0^1 F(r,r)(2r^2)\int_0^{\Delta/r} \ln(8/\xi)\, d\xi dr + O(\Delta) \tag{77a}$$

$$\simeq \int_0^1 F(r,r)\ (2r\Delta)\ln(8r/\Delta)\ dr + O(\Delta) \tag{77b}$$

The integrand factor $(2r\Delta)\ln(8r/\Delta)$ is bounded in absolute value for all combinations of r and Δ, given that both are between 0 and 1. This is clear because $x[\ln(1/x)]$ is less than $e^{-1}\simeq 0.37$. Thus, providing $F(r,r)$ is finite for all r between 0 and 1, the integral in Eq. (77b) is bounded, and it goes to zero when Δ goes to zero.

As is explained further below, it is advantageous to use basis functions $\Psi(r)$ that go to zero like $(1-r)^{1/2}$ when $r\to 1$. Such would require Ψ' to have an integrable singularity at r=1. However, the above demonstration of boundedness would no longer be valid because $F(r,s)$ would be singular when either r or s are unity. Thus, there would be a confluence of two or three singularities at the point r=s=1. To check whether this precludes the guaranteed existence of finite matrix coefficients, it is sufficient to examine the existence of the integral

$$\{\text{Contribution to } a_{jn}\} \simeq M\int_{1-\epsilon}^{1}\int_{1-\epsilon}^{1} \frac{\ln[64/(r-s)^2]}{(1-r)^{1/2}(1-s)^{1/2}}\, ds\, dr \tag{78}$$

where M is finite and ϵ is small compared to unity. The above is an integral over a small square of side length ϵ with a corner at the point r=s=1. Alternatively, it is sufficient to consider integration over the quadrant of a circle centered at r=s=1 and having radius ϵ. With the latter understanding, we let 1-r be $\alpha\epsilon\cos\phi$ and 1-s be $\alpha\epsilon\sin\phi$, such that ds dr is replaced by

$\epsilon^2\alpha\ d\alpha\ d\phi$ and α ranges from 0 to 1, while ϕ ranges from 0 to $\pi/2$. The quadrant integral corresponding to the examination exercise posed by Eq. (78) above then becomes

$$\{\text{Contribution to } a_{jn}\} \simeq M\epsilon \int_0^{\pi/2} \int_0^1 \frac{\ln\{64/[\epsilon^2\alpha^2(\cos\phi-\sin\phi)^2]\}}{(\sin\phi)^{1/2}(\cos\phi)^{1/2}}\, d\alpha\, d\phi$$

$$\simeq 4M\epsilon \ln\{8e/\epsilon\} \int_0^{\pi/4} \frac{d\phi}{(\sin\phi)^{1/2}(\cos\phi)^{1/2}}$$

$$- 4M\epsilon^2 \int_0^{\pi/4} \frac{\ln(\cos\phi-\sin\phi)\ d\phi}{(\sin\phi)^{1/2}(\cos\phi)^{1/2}} \qquad (79)$$

Both of the indicated integrations over ϕ in the latter expression exist and have values of the order of unity, even though the integrands have one and two singularities, respectively. Both terms are finite and, moreover, they go to zero when ϵ goes to zero.

One may conclude from the analysis of the present section that the integrated Green's functions may be evaluated numerically without much effort, even though the original integrals defining these quantities had integrands with singularities. Both of these integrated Green's functions have a logarithmic singularity at r=s, so when these functions appear as factors in an integrand, as in Eq. (56), the integration scheme must not explicitly ask for values at such arguments. However, the presence of these singularities does not cause any of the anticipated integrands to be nonintegrable.

10. EVALUATION OF THE MATRIX ELEMENTS

Near the edge of the disk the exact solution for the acoustic pressure on the surface must behave to leading order in the distance from the edge in the same manner as would a solution of Laplace's equation near a knife edge. Thus if α is radial distance from the edge and if ψ is angle about the edge, such that ψ is 0 on the front side and 2π on the back side, then

$$\partial^2 p/\partial\alpha^2 + \alpha^{-1}\partial p/\partial\alpha + \alpha^{-2}\partial^2 p/\partial\psi^2 \simeq 0 \tag{80}$$

for sufficiently small α. The pressure p must be finite near the edge, and it must satisfy the rigid wall boundary conditions

$$\partial p/\partial\alpha = 0 \qquad \text{at } \psi=0 \text{ and at } \psi=2\pi \tag{81}$$

This "boundary value problem" can be solved by the method of separation of variables, with the result that, to leading order in α,

$$p \simeq A + B\,\alpha^{1/2}\cos(\psi/2) \tag{82}$$

where A and B are "constants". For the particular case of sound generated by a disk in rigid body transverse vibration, symmetry and continuity require that p be zero at the edge, so the "constant" A is identically zero. Hence, on the front surface, where $\psi=0$, the implication of the above equation is

$$p \sim (\text{distance from edge})^{1/2} \tag{83}$$

near the edge of the disk.

The above deduced behavior must be exhibited by the exact solution for p of either the differential integral equation or of the variational principle. It is not, however, a requirement that must be imposed at the outset on the trial functions or the basis functions. The derived approximate solution, if the number of basis functions is sufficiently large, can be expected, in the aggregate, to approximate the behavior of Eq. (83).

Since we know the result (83) in advance of a detailed numerical solution, a strong argument can be made that it should be incorporated into the trial functions at the outset, for then a faster convergence toward the exact solution might be achieved. One possibility for doing this is to include a factor $(1-r^2)^{1/2}$ in each of the $\Psi_n(r)$. One could take, for example, $\Psi_1(r)=(1-r^2)^{1/2}$, which turns out to be the exact result [Pierce, 1981] for the pressure distribution on the disk in the limit $ka \to 0$. Doing so, however, introduces another singularity into one of the integrands of the integrals that define the matrix elements a_{jn} in Eq. (56). Clearly, $\Psi_1'(r)$ is infinite at r=1. The singularity is integrable, but integration over singular integrands is an inherent source of numerical difficulties. Such difficulties are often surmountable with a change of integration variable that transforms the original integrand into one in which the singularity does not appear.

With the purpose just described in mind, we denote the desired transformation as

$$r = g(u); \qquad s = g(w) \tag{84}$$

Then

$$\Psi_m'(r)\, dr = \frac{d}{du}\, \Psi_m[g(u)]\, du \tag{85}$$

If g(u) is chosen appropriately, then $(d/du)\Psi_n(g(u))$ will be finite at the value of u where g(u)=1, even if $\Psi_m'(1)$ is not. It is convenient to have the limits of the domain of u and w match those of r and s, so we introduce the requirement that

$$g(0) = 0; \quad g(1) = 1 \tag{86}$$

The choice for the integration variable transformation function g could be different for different basis functions, but it is advantageous for us to require that it be the same for all elements within the same set, such that no symmetry properties are lost; this is assumed to have been done in the discussion that follows.

In the case of the basis function $\Psi_1=(1-r^2)^{1/2}$, we shall set

$$g(u) = \sin\left[\frac{\pi u}{2}\right] \tag{87}$$

This satisfies Eqs. (86), and the derivative in Eq. (85) transforms to

$$\frac{d}{du}\,\Psi_1[g(u)] = \frac{d}{du}\left[1-\sin^2\left[\frac{\pi u}{2}\right]\right]^{1/2}$$

$$= -\frac{\pi}{2}\sin\left[\frac{\pi u}{2}\right] \tag{88}$$

which is well-behaved for all u.

To simplify the notation, we let $\Gamma_n(u)$ denote the n-th basis function when it is regarded as a function of the alternate integration variable u, so that

$$\Gamma_n(u) \equiv \Psi_n[g(u)] \tag{89}$$

Then substitution of Eqs. (84), (85), and (89) into Eq. (56) yields

$$a_{jn} = a_{nj} = (ka)^2 \int_0^1\int_0^1 g'(u)g'(w)\Gamma_j(u)\Gamma_n(w)C_1(g(u)|g(w))\,dwdu$$

$$- \int_0^1\int_0^1 \Gamma_j'(u)\Gamma_n'(w)C_1(g(u)|g(w))\,dwdu \tag{90}$$

Because the source variable r and field variable s undergo the same transformation, the singularity contributed by the integrated Green's functions continues to lie on the diagonal, which is now the line w=u. This line can be taken to be a boundary line for the integration domain, by using the simple identity

$$\int_0^1 \int_0^1 F(u,w)\, dwdu = \int_0^1 \int_0^u \Big[F(u,w) + F(w,u)\Big] dw\, du \tag{91}$$

where the function F(u,w) is arbitrary. With the recognition that the integrated Green's functions are unchanged by interchange of their arguments, the application of this identity to Eq. (90) yields

$$a_{jn} = (ka)^2 \int_0^1 \int_0^u \Big[\Gamma_j(u)\Gamma_n(w)+\Gamma_j(w)\Gamma_n(u)\Big] g'(u)g'(w)\; C_1(g(u)|g(w))dw\, du$$

$$- \int_0^1 \int_0^u \Big[\Gamma_j'(u)\Gamma_n'(w)+\Gamma_j'(w)\Gamma_n'(u)\Big]\; C_2(g(u)|g(w))dw\, du \tag{92}$$

In a similar manner, the "forcing vector" components b_n, given previously by Eq. (57), when expressed in terms of the new variable of integration u, become

$$b_n = 2\pi ika \int_0^1 g(u)g'(u)\Gamma_n(u)\, du \tag{93}$$

The integrations required to evaluate the components b_n may be effected by direct numerical quadrature; Simpson's 1/3 rule should be adequate. The interval $0 \le u \le 1$ is regarded as composed of J intervals, where J is even; then

$$b_n = 2\pi ika \sum_{j=0}^{J} \Gamma_n(u_j)g(u_j)g'(u_j)\nu_j \tag{94}$$

where $u_j = j/J$ is the endpoint of the j-th integration interval. The weighting coefficients ν_j for Simpson's "1/3" rule are 1/(3J), 4/(3J), 2/(3J), 4/(3J), ..., 4/(3J), and 1/(3J) for j = 0, 1, 2, 3,....., J-1, and J. The J-th term in the above sum is identically zero because, as remarked previously, one must have $\Gamma_n(1) = 0$. Recall that all admissible basis functions must vanish at the edge of the disk.

The double integrations required for the evaluation of the matrix coefficients a_{jn} in Eq. (90) cannot be evaluated by sequential application of a one-dimensional integration rule. Such an approach, which would discretize u and w in the same manner, would tacitly assume that the integrand be finite along the line w=u (where, in fact, the integrand is singular), since the integration algorithm would use values of the integrand evaluated at points along this line. We circumvent this difficulty by using the alternative expression, Eq. (92), and by employing an area integration rule based on interior points. As shown in Fig. 4, the triangular integration domain, which is bounded by the three lines w=0, u=1, and w=u, is segmented into K(K-1) slanted squares, and 2K triangles, based on K divisions along the lines w=0 and u=1. The value of K need not be the same as the number of intervals J that are used in the evaluation of the b_n. For integration over each of the interior squares, one uses a nine-point integration rule [Davis and Polonsky, 1972], such that the integral over a square is approximated by the area of the square times a weighted average of the values of the integrand at nine interior points. The locations of these points and the weighting factors are selected such that the integration will be exact if the integrand is a polynomial of the fifth degree. The relative location of these interior points within a single square is shown in Fig. 5.

Let $2\Delta = 1/K$ be the width of a subdivision interval along the line w=0 between u=0 and u=1. Each of the interior squares will then have length $2^{1/2}\Delta$ on a side, diagonals of length 2Δ, and areas $2\Delta^2$. The centerpoints of these interior squares are at

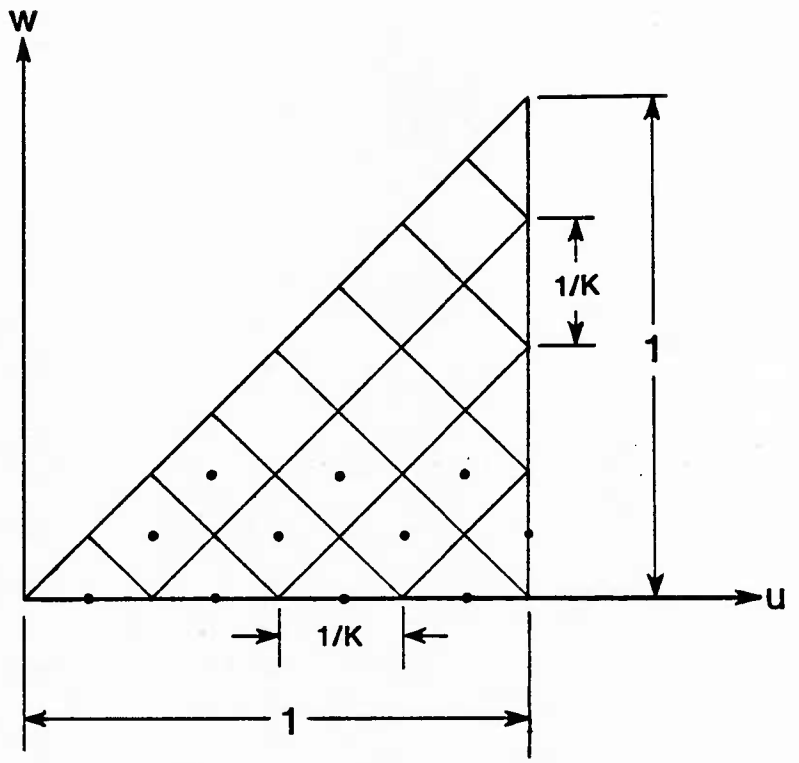

Figure 4. Discretization of integration domain into segments.

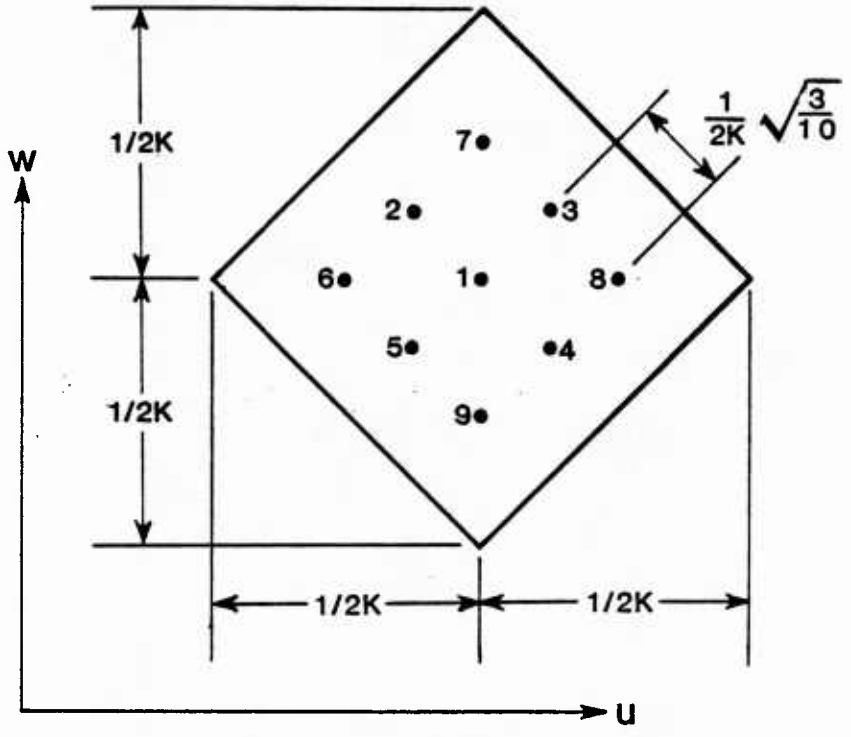

Figure 5. Numerical integration points for a square segment.

$$u_{km,c} = 2k\Delta + (m-1)\Delta; \quad w_{km,c} = m\Delta \quad \begin{cases} m = 1, \ldots, 2K-2k \\ k = 1, \ldots, K-1 \end{cases} \tag{95}$$

such that squares distinguished by the same value of the index k have center-points that lie along a common line parallel to the diagonal line w=u; squares with the same k are numbered by the index m in the order in which they are encountered as one moves up this line toward increasing u and w. The coordinates of the nine integrand sampling points within a given square and the corresponding weights (which sum to unity) are

$$u_{km}(q) = u_{km,c} + a(q)\Delta; \quad w_{km}(q) = w_{km,c} + b(q)\Delta; \; q = 1, 2, .., 9$$

$$a(1) = 0 \qquad b(1) = 0$$

$$a(2) = -(3/20)^{1/2} \qquad b(2) = +(3/20)^{1/2}$$

$$a(3) = +(3/20)^{1/2} \qquad b(3) = +(3/20)^{1/2}$$

$$a(4) = +(3/20)^{1/2} \qquad b(4) = -(3/20)^{1/2}$$

$$a(5) = -(3/20)^{1/2} \qquad b(5) = -(3/20)^{1/2}$$

$$a(6) = -(3/5)^{1/2} \qquad b(6) = 0$$

$$a(7) = 0 \qquad b(7) = +(3/5)^{1/2}$$

$$a(8) = +(3/5)^{1/2} \qquad b(8) = 0$$

$$a(9) = 0 \qquad b(9) = -(3/5)^{1/2}$$

$$\nu_{sq}(1) = 16/81; \quad \nu_{sq}(2) = \nu_{sq}(3) = \nu_{sq}(4) = \nu_{sq}(5) = 10/81$$

$$\nu_{sq}(6) = \nu_{sq}(7) = \nu_{sq}(8) = \nu_{sq}(9) = 25/324 \tag{96}$$

The integrations over the K triangles that fill the gaps left by the squares along the line w=0 (lower edge, abbreviated le) can be similarly performed using a seven-point integration rule. The lower edges of these triangles have their centers at u = Δ, 3Δ, 5Δ, ..., $(2K-1)\Delta$; each has height Δ and area Δ^2. The coordinates of the integrand sampling points within the k-th triangle and the corresponding weights are derived by mapping the 45^o-45^o-90^o

triangle to an equilateral triangle and then using a standard integration scheme for the latter. [One of the expressions on page 483 in the Dover edition of Abramowitz and Stegun's Handbook of Mathematical Functions has an error. What appears there as $-(15)^{1/2}+1$ should be $-[(15)^{1/2}+1]$.]The result of such a derivation yields

$$u_{le,k}(q) = (2k-1)\Delta + \alpha(q)\Delta; \quad w_{le,k}(q) = \beta(q)\Delta; \; q = 1, 2, .., 7$$

$$\alpha(1) = 0 \qquad \beta(1) = 1/3$$

$$\alpha(2) = 0 \qquad \beta(2) = [9 + 2(15)^{1/2}]/21$$

$$\alpha(3) = -[1 + (15)^{1/2}]/7 \qquad \beta(3) = [6 - (15)^{1/2}]/21$$

$$\alpha(4) = +[1 + (15)^{1/2}]/7 \qquad \beta(4) = [6 - (15)^{1/2}]/21$$

$$\alpha(5) = 0 \qquad \beta(5) = [9 - 2(15)^{1/2}]/21$$

$$\alpha(6) = -[(15)^{1/2}-1]/7 \qquad \beta(6) = [6 + (15)^{1/2}]/21$$

$$\alpha(7) = +[(15)^{1/2}-1]/7 \qquad \beta(7) = [6 + (15)^{1/2}]/21$$

$$\nu_{tri}(1) = \frac{270}{1200}; \quad \nu_{tri}(2) = \nu_{tri}(3) = \nu_{tri}(4) = \frac{155 - (15)^{1/2}}{1200}$$

$$\nu_{tri}(5) = \nu_{tri}(6) = \nu_{tri}(7) = \frac{155 + (15)^{1/2}}{1200} \qquad (97)$$

Similarly, for the K triangles with hypotenuses along the line u=1 (right edge, abbreviated re), one has

$$u_{re,k}(q) = 1 - \beta(q)\Delta; \quad w_{re,k}(q) = (2k-1)\Delta + \alpha(q)\Delta; \; q = 1, 2, .., 7 \qquad (98)$$

with the $\alpha(q)$, $\beta(q)$, and $\nu_{tri}(q)$ the same as in Eq. (97).

The integration scheme just described gives the following approximate representation for the double integral of any function F(u,w) over the unit square:

$$\int_0^1 \int_0^1 F(u,w)\, dwdu = \int_0^1 \int_0^u \Big[F(u,w) + F(w,u)\Big] dw\, du$$

$$= \sum_{k=1}^{K-1} \sum_{m=1}^{2K-2k} \sum_{q=1}^{9} 2\Delta^2\, S(u_{km}(q),\, w_{km}(q))\, \nu_{sq}(q)$$

$$+ \sum_{k=1}^{K} \sum_{q=1}^{7} \Delta^2\, S(u_{le,k}(q),\, w_{le,k}(q))\, \nu_{tri}(q)$$

$$+ \sum_{k=1}^{K} \sum_{q=1}^{7} \Delta^2\, S(u_{re,k}(q),\, w_{re,k}(q))\, \nu_{tri}(q) \qquad (99)$$

where

$$\Delta = 1/(2K); \qquad S(u,w) = F(u,w) + F(w,u) \qquad (100)$$

For the computation of a particular a_{jn}, the function $F(u,w)$ is identified as

$$\text{Integrand } F(u,w) \text{ for } a_{jn} = (ka)^2 \Gamma_j(u)\Gamma_n(w)g'(u)g'(w)\, C_1(g(u)|g(w)) - \Gamma_j'(u)\Gamma_n'(w)\, C_2(g(u)|g(w)) \qquad (101)$$

Many numerical operations are required to evaluate the matrix elements a_{jn} using Eqs. (99-101) for all combinations of j and n. One saving comes from the recognition that the sampled values of the integrated Green's functions appearing there are independent of the (j,n) pair under consideration. Hence, the values of $g'(u)g'(w)C_1(g(u)|g(w))$ and $C_2(g(u)|g(w))$ for all points (u,w) used in the sums in Eq. (99) may be stored in arrays, and then recalled whenever necessary to calculate the integrand (101) at the corresponding points.

11. FINITE ELEMENT BASIS FUNCTIONS

The implication thus far has been that the set of basis functions consists of a few analytical functions satisfying the boundary condition $\hat{p}(1)=0$. However, the generic trial functions will also be admissible if the basis functions are not smooth curves, but merely continuous, with a finite number of derivative discontinuities.

Suppose one seeks to represent the surface pressure distribution $\hat{p}(r)$ by N discrete values at evenly spaced radial distances $r_n=(n-1)/N$. As shown in Fig. 6, a linear interpolation between such discrete values may be represented as a superposition of piecewise linear functions. Let $\Psi_n(r)$ denote a linear finite element basis function centered at r_n. We define this basis function to be unity at $r=r_j$, and to vanish beyond the adjacent points r_{n-1} and r_{n+1}. A simple algorithm for evaluating these basis functions defines the local distance

$$d = r - r_n = r - (n-1)\Delta; \quad \Delta = 1/N \tag{102}$$

and then sets

$$\Psi_n(r) = \begin{cases} 1-|d|/\Delta & \text{if } |d|<\Delta \\ 0 & \text{if } |d|>\Delta \end{cases}$$

$$\Psi_n'(r) = \begin{cases} -1/\Delta & \text{if } 0\le d<\Delta \\ 1/\Delta & \text{if } -\Delta<d<0 \\ 0 & \text{if } |d|>\Delta \end{cases} \tag{103}$$

When we use these finite element basis functions to form a generic trial function as in Eq. (54), it is immediately apparent that the coeficients of the basis functions are identical to the discrete pressure values, that is,

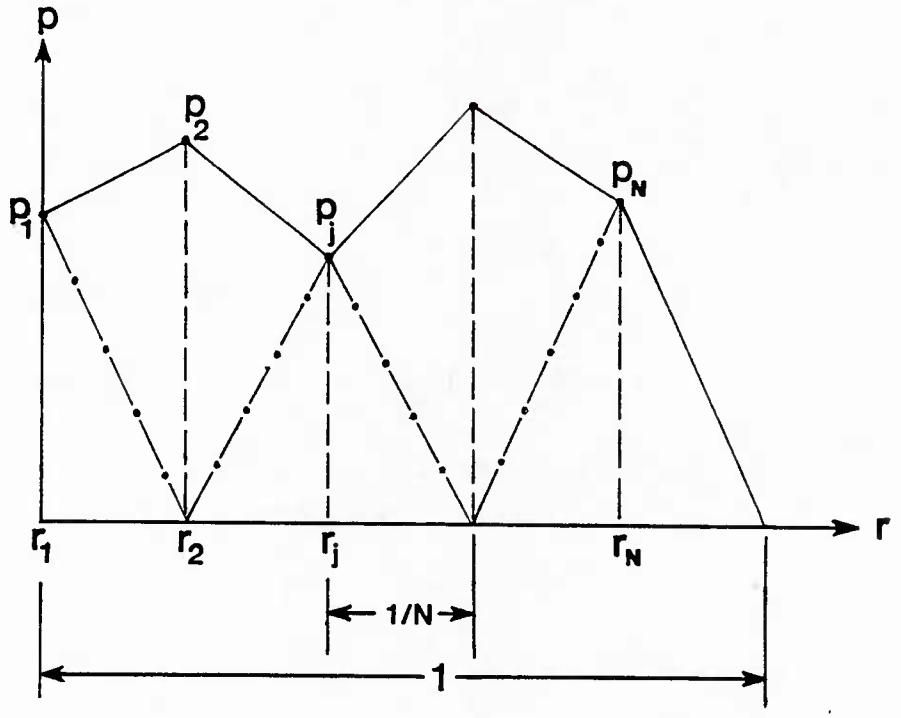

Figure 6a. Linear interpolation of a discretized pressure.

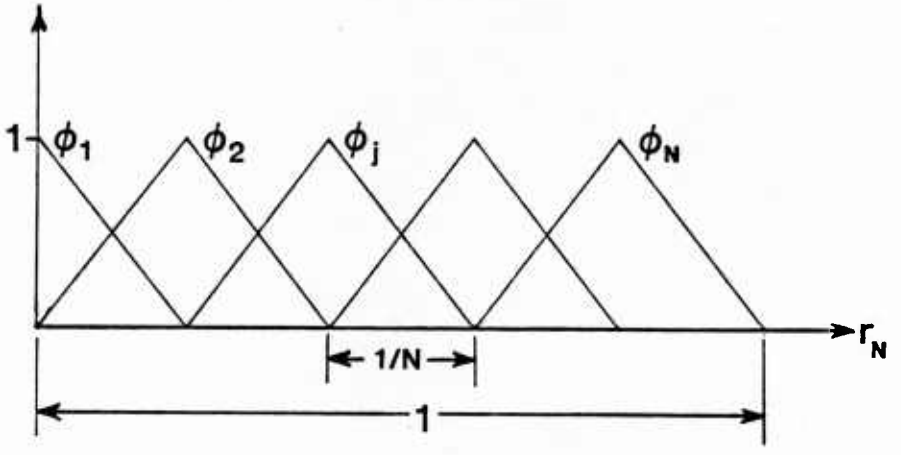

Figure 6b. Finite element basis functions.

$$P_n = \hat{p}(r_n) \qquad \text{where} \quad r_n = (n-1)\Delta \tag{104}$$

In principle, we could employ the finite element basis functions directly using the general formulation and numerical scheme developed in the preceding section. However, doing so would lose potential computational savings resulting from the fact that the finite element basis functions vanish over a large portion of the range $0 \le r \le 1$. Such savings may be very significant, because many basis functions may be required for an accurate description of the pressure distribution.

For such a purpose, it is first convenient to split the overall sum of the type in Eq. (99) for the evaluation of a given a_{jn} using Eqs. (100) and (101) into two partial sums S_{jn} and S_{nj}, such that

$$a_{jn} = a_{nj} = S_{jn} + S_{nj} \tag{105}$$

where

$$\begin{aligned} S_{jn} = & \sum_{k=1}^{K-1} \sum_{m=1}^{2K-2k} \sum_{q=1}^{9} 2\Delta^2 \, F_{jn}(u_{km}(q),\, w_{km}(q)) \, \nu_{sq}(q) \\ & + \sum_{k=1}^{K} \sum_{q=1}^{7} \Delta^2 \, F_{jn}(u_{le,k}(q),\, w_{le,k}(q)) \, \nu_{tri}(q) \\ & + \sum_{k=1}^{K} \sum_{q=1}^{7} \Delta^2 \, F_{jn}(u_{re,k}(q),\, w_{re,k}(q)) \, \nu_{tri}(q) \end{aligned} \tag{106}$$

$$\begin{aligned} F_{jn}(u,w) = & \; (ka)^2 \Gamma_j(u)\Gamma_n(w)g'(u)g'(w) \, C_1(g(u)|g(w)) \\ & - \Gamma_j'(u)\Gamma_n'(w) \, C_2(g(u)|g(w)) \end{aligned} \tag{107}$$

Since the sum in Eq. (106) uses only points for which the w coordinate is less than the u coordinate, the sum defining the S_{jn} will be identically zero if all of the u's for which $\Gamma_j(u)$ is nonzero are less than all of the u's for which $\Gamma_n(u)$ is nonzero.

By definition, $\Gamma_j(u) = \Psi_j(g(u))$. For the finite element basis functions defined by Eqs. (103), the products $\Psi_j(r)\Psi_n(s)$ and $\Psi_j(r)\Psi_n'(s)$ are nonzero only in the domain R_{jn} defined by

$$R_{jn}: \qquad 0 \le \frac{j-2}{N} \le r \le \frac{j}{N}, \qquad 0 \le \frac{n-2}{N} \le s \le \frac{n}{N} \tag{108}$$

This domain is depicted in Fig. 7a. Because the boundaries of R_{jn} correspond to constant values of r or s, the transformation of r=g(u) and s=g(w) maps r_{jn} into the region U_{jn} in the u-w plane, as depicted in Fig. 7b. This region is defined by

$$U_{jn}: \quad 0 \le g^{-1}\left(\frac{j-2}{N}\right) \le u \le g^{-1}\left(\frac{j}{N}\right); \qquad 0 \le g^{-1}\left(\frac{n-2}{N}\right) \le w \le g^{-1}\left(\frac{n}{N}\right) \tag{109}$$

where g^{-1} denotes the inverse transformation.

Figure 7b also depicts the region covered by several integration area-elements (i.e., those squares and triangles used in the breaking up of the overall integration domain in the u-w plane). Any area-element that does not overlap the finite element domain U_{jn} may be skipped in the formation of the partial sum S_{jn}. Furthermore, when an area element only partially overlaps U_{jn}, the contributions from integrands evaluated at points (u,w) that do not satisfy Eq. (109) may also be skipped. This observation reduces the number of computations because it avoids operations on terms that would eventually be found to vanish.

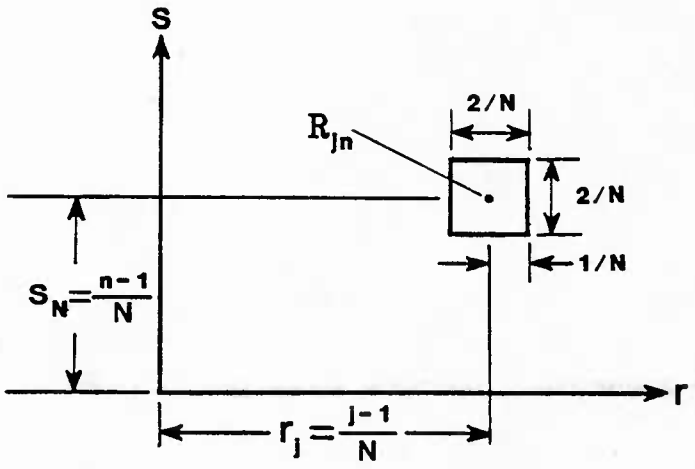

Figure 7a. Domain of a finite element in physical variables.

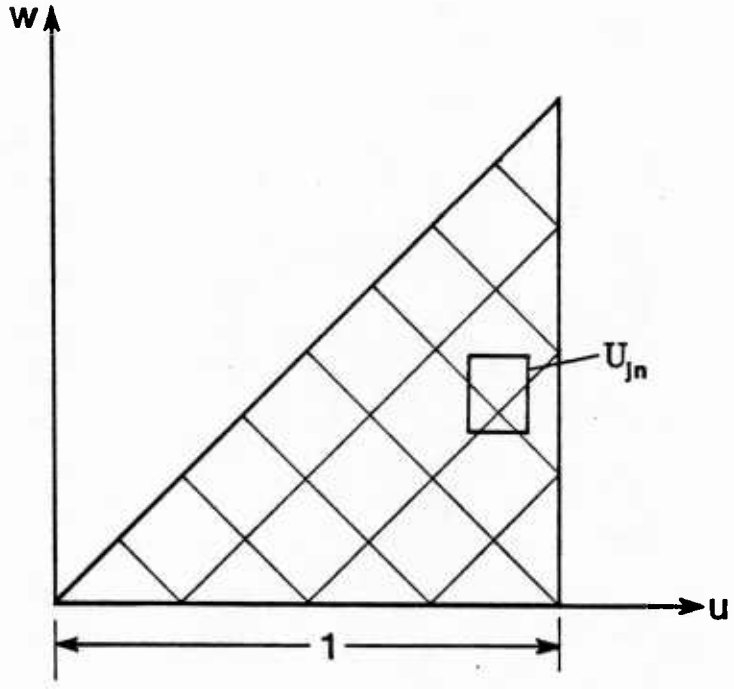

Figure 7b. Domain of a finite element in integration variables.

Because Eqs. (105), (106), and (107) are equivalent to Eqs. (99), (100), and (101), we may also employ the partial sums to evaluate the coefficients a_{jn} corresponding to distributed basis functions. Hence, the only difference in the algorithms for distributed and finite element basis functions is that the latter should be checked for overlap according to Eq. (109). We also implemented an analogous overlap check in the computation of the summation for b_n, Eq. (94), in the case of finite element basis elements.

12. NUMERICAL RESULTS

An analytical solution for the pressure distribution along the face of a thin disk is available for the limiting case of very low frequencies, $ka \to 0$. Thus, we chose this case for the first validation of the variational principle and its implementation. When the pressure on the positive z face is $v_o \exp(-i\omega t)$, with $ka \ll 1$, then the pressure is

$$p = -\frac{2}{\pi} i\rho c v_o ka(1 - r^2)^{1/2} \tag{110}$$

In addition to providing a simple expression for comparison with numerical results, this form provides an indirect method of verification. If this mode is used as one of several modes, the amplitudes of the others should be very small when $ka \ll 1$. We tested the accuracy of this hypothesis by redefining the modal amplitudes to be P_j/ka and selecting the modes to be

$$\Psi_j = r^{2(j-1)}(1 - r^2)^{1/2} \tag{111}$$

Table 1 gives the modal amplitudes as a function of the number of modes N in the series expansion. These results were developed by applying the sine transformation for the radial distance, Eq. (87), with the number of integration segments set at K = 10. The accuracy of the variational principle is remarkable. Note that the first mode is very close to the value in Eq. (110), while the other modes are three orders of magnitude smaller.

Table 1

Approximate solutions at ka → 0	Coefficients C_n corresponding to the shape functions $r^{2(n-1)}\sqrt{1-r^2}$, n = 1, 2, ... N		
	C_1	C_2	C_3
N = 1	-0.63875		
N = 2	-0.63875	0.27369×10^{-6}	
N = 3	-0.63875	-0.21376×10^{-5}	0.27127×10^{-5}
Analytical solution at ka → 0	-0.63662		

Rayleigh-Ritz approximations of the acoustic pressure on the surface of a transversely vibrating rigid circular disk. The number of integration intervals is set at 20 in each case.

We also checked the finite element results for ka→0 against Eq. (110). For this, we must consider the general effect of inadequate choices for the number of elements N and the number of integration segments K. The value of N must be sufficiently large to represent the pressure as a sequence of discrete values. We estimated that $N \geq 5$ was appropriate for the pressure given by Eq. (110). When the number of elements has been set, the value of K must be sufficient to place several integration points within the domain covered by a finite element. We found that sensible results are obtained if $K \geq N$.

The transformation in Eq. (87) was used in the evaluation of the finite elements model, although it is not necessary to do so. We will discuss the reasons for this choice later. Comparison of finite element predictions for several values of K when N = 5 and 10 are shown in Figure 8 and 9, respectively. The agreement between the finite element formulation and the analytical solution is evident. Clearly, ten elements provide better resolution of the pressure distribution, but the results exhibit substantial numerical noise with increasing K.

Such noise could have two causes. The computations were performed on a VAX 11/750, which due to hardware limitations in its current configuration slows drastically when performing double precision complex arithmetic in programs that access virtual memory. However, we checked the results with comparable calculations on a CDC CYBER 785, which doubles the arithmetic precision because the word size on the CYBER is twice as large as that on the VAX. Nevertheless, we found that the results changed very little. We have concluded that the primary source of error generation for finite elements is the usage of the piecewise linear modes in Eqs. (103). As the interval covered by each element decreases, continuous analytical derivatives are modeled in those modes as large values that change sign at the center point. When ka = 0, the integrand forming the coefficient array [A] depends solely on these derivatives, so it is not surprising to see the predictions display numerical noise. We believe that these problems would be ameliorated through the application of smoother finite elements, such as polynomials.

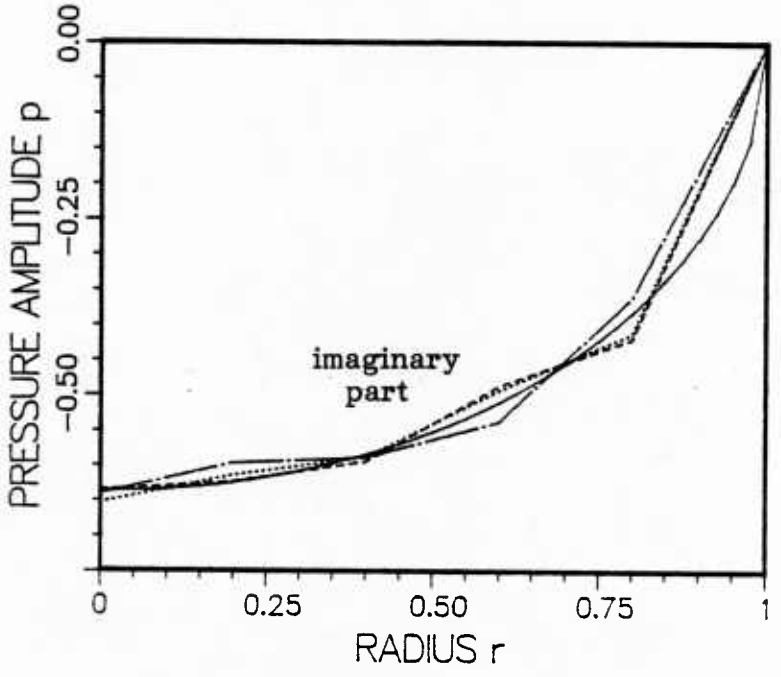

Figure 8. Pressure distribution when ka=0 using five finite elements (N=5). —·—: K=5; ·····: K=10; − − −: K=20; —— : Eq. (110).

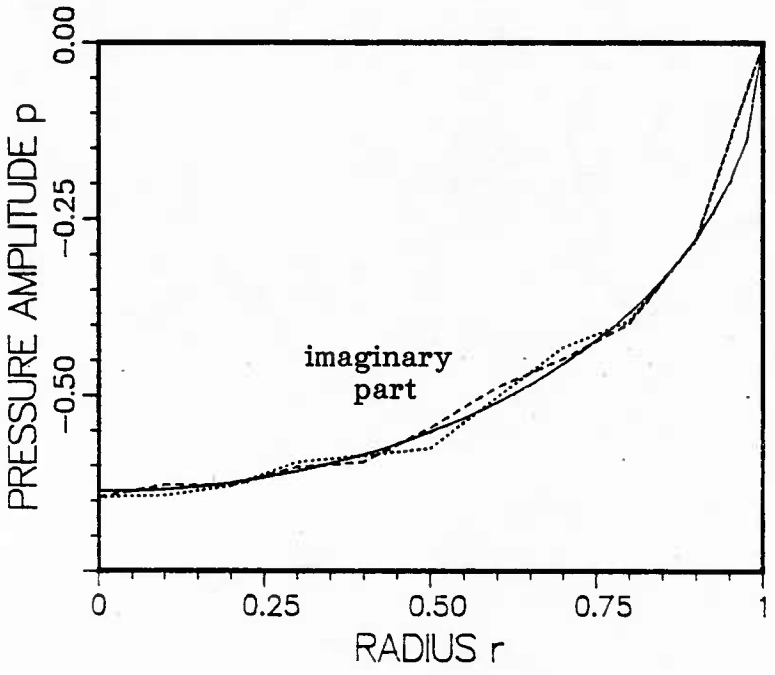

Figure 9. Pressure distribution when ka=0 using ten finite elements (N=10). ····: K=10; – – –: K=20; ——: Eq. (110).

The modes in Eq. (111) are such that one matches the analytical solution. A fairer test is the ability of the variational principle to predict the correct pressure distribution when the modes that are selected merely satisfy the boundary condition that p = 0 at the edge. Our choice for this evaluation was a half-range cosine series,

$$\Psi_j = \cos\left[\left((2j-1)\frac{\pi r}{2}\right]\right. \tag{112}$$

Figure 10 shows that an expansion using five of these modes compares quite favorably with Eq. (110). Indeed, the prediction is virtually identical to a Fourier series expansion of Eq. (110) using Eq. (112).

The results in Figure 10 were computed using the transformation in Eq. (87). Such a transformation is not required for these functions, nor for the finite element modes. Both Eq. (112) and Eq. (103) give finite values for $\dot{\Psi}_j$ at r=1. Hence it would be possible in either case to perform the numerical integrations leading to [A] with an identity transformation of g(u)=1.

When we employed the identity transformation, we found that favorable comparison with the analytical solution required more modes or finite elements. This observation has a ready explanation. The integration scheme described by Figures 4 and 5 uses a uniform mesh in the u-w plane. The transformation in Eq. (87) maps these points into a nonuniform mesh in the r-s plane, such that the density of integration points increases monotonically as $r\to1$ and $s\to1$. Thus, the transformation gives a better description of, and greater emphasis on, the behavior near the edge, where diffraction effects are most significant.

Based on these observations, we decided to perform all evaluations of the pressure distribution for non-zero ka using Eq. (87). Also, the nature of the slope singularity at r=1 is not altered when $ka\neq0$. For this reason, further evaluations using analytical modes shall only be based on Eq. (111).

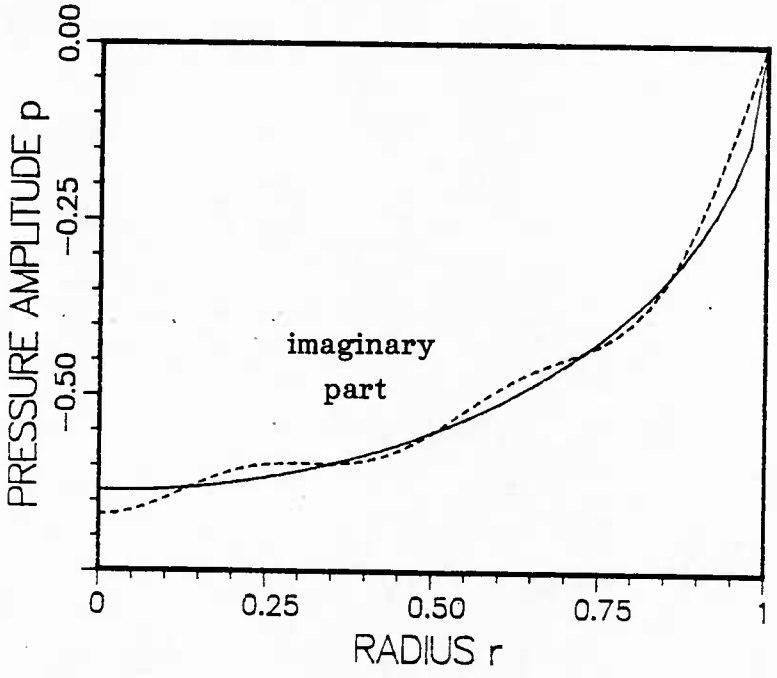

Figure 10. Pressure distribution when ka=0 using sinusoidal nodes: -----: N=5 & K=10; ——: Eq. (110).

Data with which the results of the variational principle may be compared is somewhat sparse. Fortunately, the problem of diffraction of sound by a circular disk is analogous to the present radiation problem. Consider a plane wave at normal incidence to a thin circular disk that is fixed in space. Let p_s denote the scattered acoustic wave, and let the z axis coincide with the axis of symmetry. For a wave incident on the face $z=0^+$, corresponding to propagation in the direction of decreasing z, the total acoustic pressure is

$$p = [p_s(r,z) + p_o \exp(-ikaz)]\exp(-i\omega t) \tag{113}$$

where r and z are nondimensional cylindrical coordinates and p_o is the amplitude of the incident wave.

Since the disk is stationary, the acceleration of a fluid particle must vanish at z=o, which leads to

$$\left.\frac{dp_s}{dz}\right|_{z=0} = ika\, p_o \tag{114}$$

For comparison, we find in Eq. (4) that the boundary condition for acoustic radiation from a disk is

$$\left.\frac{\partial p}{\partial z}\right|_{z=0} = i\omega\rho v_o \tag{115}$$

when the disk velocity is $v_o \exp(-i\omega t)$. Comparison of Eqs. (114) and (115) shows that results for scattering may be applied to the radiation problem by letting $p_o = \rho c v_o$.

The diffraction problem was solved by Leitner [1949] using oblate spheroidal wave functions. Results from that analysis were obtained for ka=1,2,3,4, and 5, so those are the cases we shall consider here. When $ka \to 0$, the pressure amplitude is a negative imaginary quantity, corresponding to pressure that is in phase with the acceleration, in other words, a virtual mass impedance. When $ka > 0$, the pressure has both real and imaginary parts.

Figures 11 - 15 compare the results for the analytical modes in Eq. (111) with those obtained by Leitner. The analytical mode results were obtained for series truncation at N=5, while 10 finite elements were used.

The analytical modes give excellent predictions. Indeed, the slope discontinuity at the edge seems to be modeled better by the variational principle. Another highlight of Figures 13 and 14 is that they show that the earlier results for ka = 1 and 2 were interchanged.

The finite element model seems to work reasonably well. However the accuracy is not as good as that obtained from the analytical modes. One might think that the discrepancies are due to the relatively small number of elements and integration points. Figure 16 shows the pressure distribution for ka=5 using more elements and a finer integration mesh. Little improvement is obtained by increasing the number of integration points, whereas increasing the number of elements increases the numerical noise associated with the piecewise linear nature of the elements.

The numerous computer runs we made led us to inquire about the relationship between the execution time and the values of K and N. We derived an estimate from the recognition that the bulk of the computations are associated with three distinct operations:

(1) evaluation of the integrated Green's functions $C_j(g(u_{kl})|g(w_{kl}))$,
(2) formation of the coefficients a_{jn},
(3) solution of the simultaneous equations to find the modal amplitudes.

It is clear from Figure 4 that the number of integration points is proportional to K^2, so the time required for phase 1 may be expected to have a comparable dependency. Note that the number of modes is irrelevant for this task. For the second phase, Eqs. (106) and (107) indicate that each coefficient a_{jn} is obtained from a summary over K^2 terms. Since the number of these coefficients is proportioned to N^2, we expect that the time in the second phase will be proportional to N^2K^2. Finally, the time required to

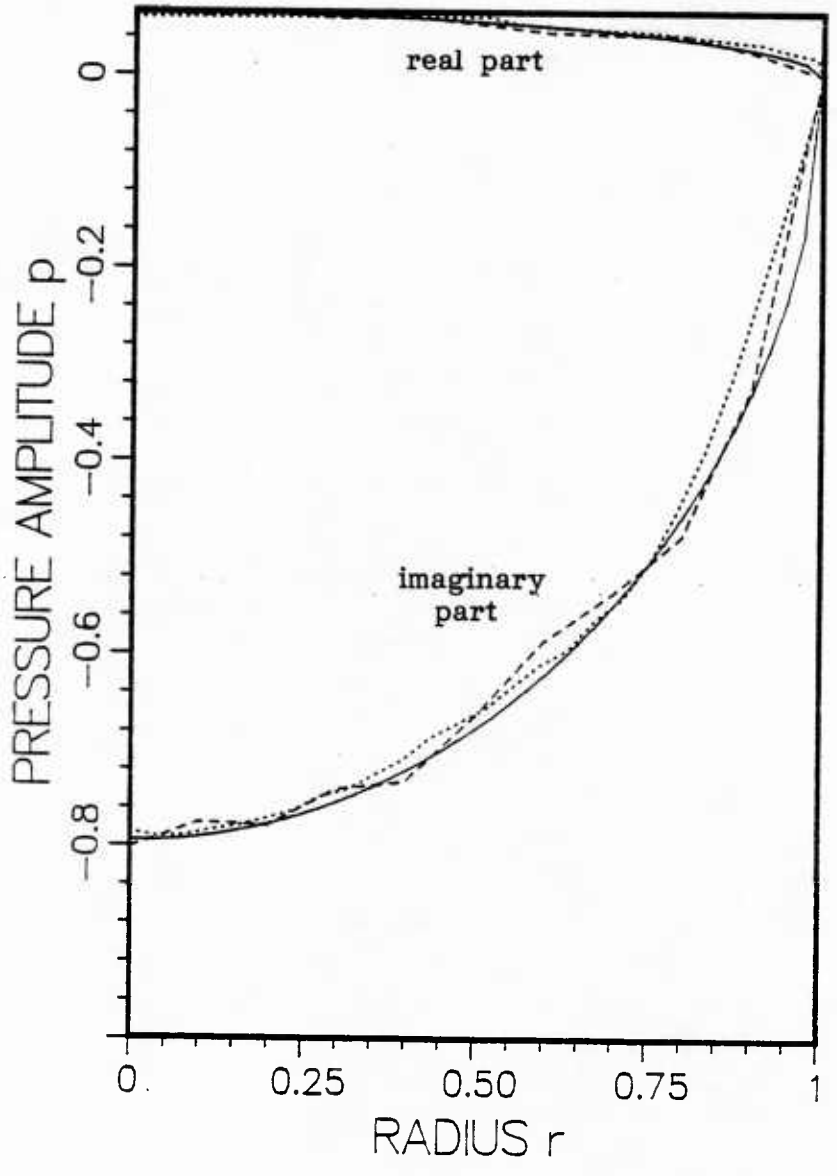

Figure 11. Pressure distribution when ka=1.

——: analytical modes (N=5, K=10);

– – –: finite element modes (N=10, K=20);

·····: Leitner [1949].

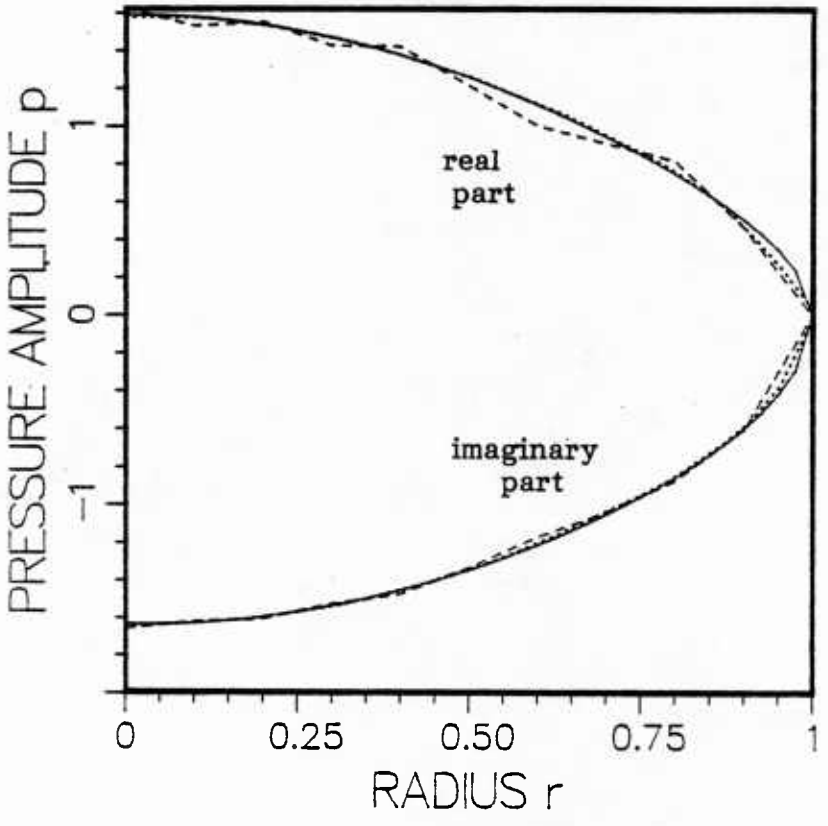

Figure 12. Pressure distribution when ka=2.

——: analytical modes (N=5, K=10);

– – –: finite element modes (N=10, K=20);

·····: Leitner [1949].

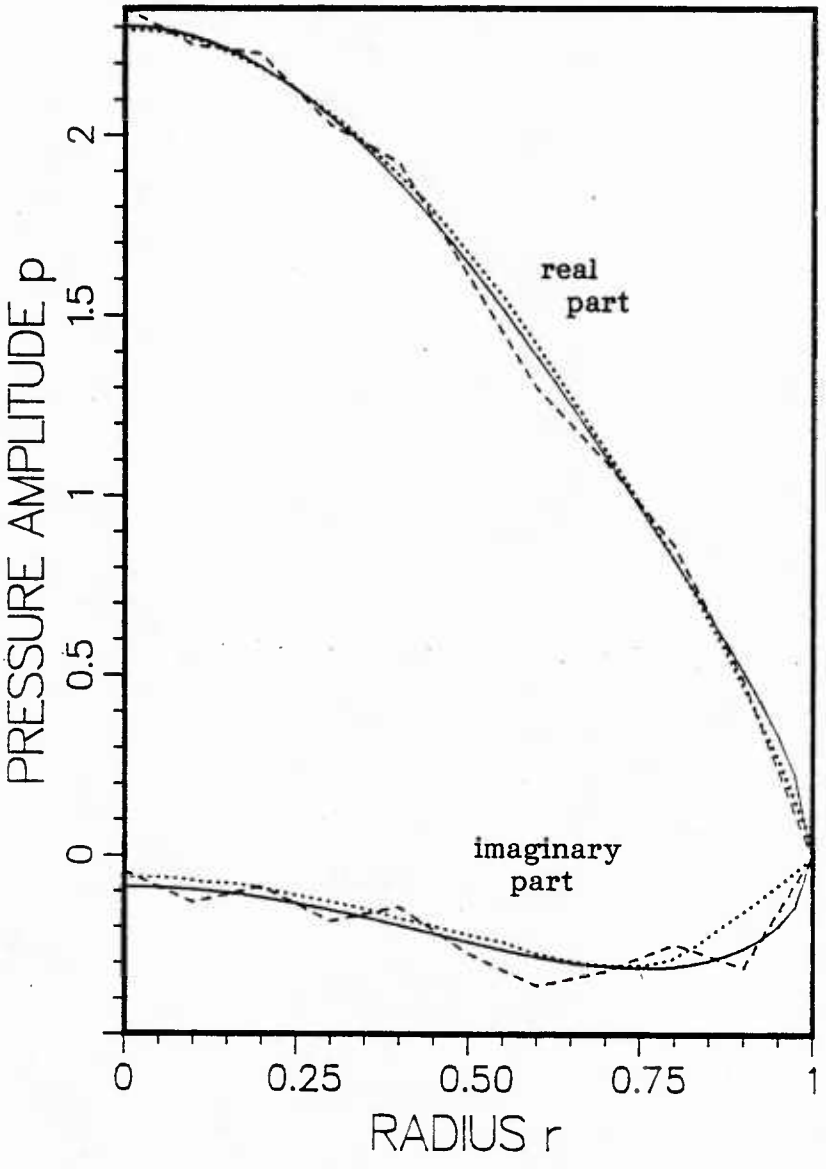

Figure 13. Pressure distribution when ka=3.
——: analytical modes (N=5, K=10);
– – –: finite element modes (N=10, K=20;
·····: Leitner [1949].

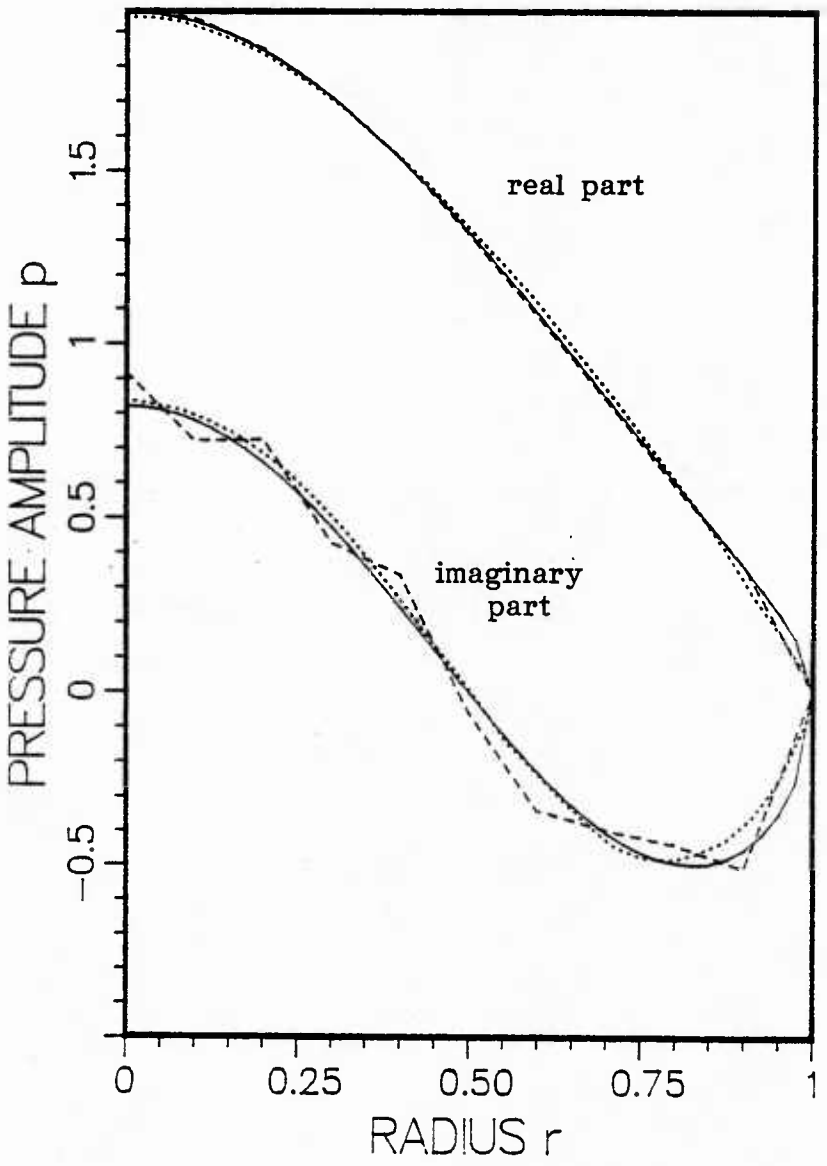

Figure 14. Pressure distribution when ka=9.

——: analytical modes (N=5, K=10);

– – –: finite element modes (N=10, K=20;

·····: Leitner [1949].

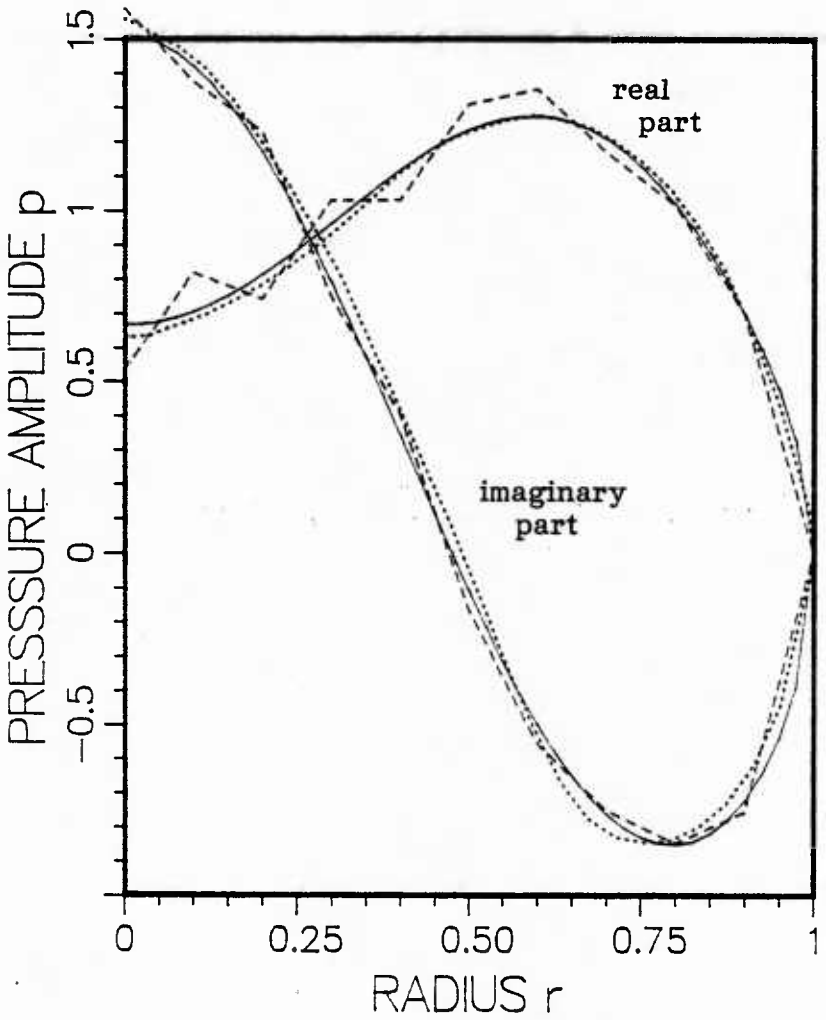

Figure 15. Pressure distribution when ka=5.
——: analytical modes N=5, K=10);
– – –: finite element modes (N=10, K=20;
·····: Leitner [1949].

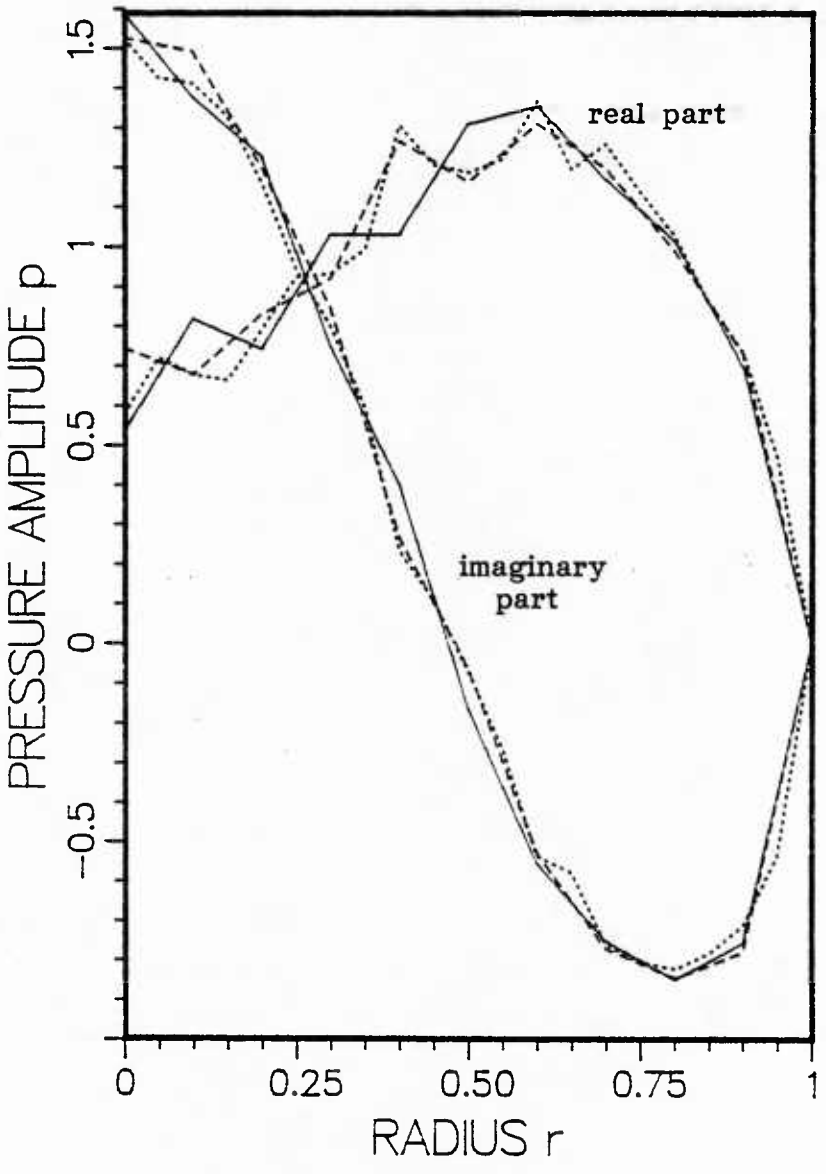

Figure 16. Influence of number of modes and integration points using finite elements, ka=5. ——: N=10 & K=20); – –; N=10 & K=40; ·····: N=20 & K=40.

solve the system equations is independent of K. The complex equation solver we employed (LEQ2C in the IMSL library) seems to require N^2 operation.

These heuristic evaluations led us to expect that the computation time would fit $T=C_1K^2+C_2N^2K^2+C_3N^2$. Reasonably close fits for the runs we made on a VAX 17/750 (with a floating point processor but no array processor) are

analytical modes:

$$T(\text{sec}) \approx 0.75K^2 + 0.01K^2N^2 + 0.14N^2$$

finite elements

$$T(\text{sec}) \approx 0.66\ K^2 + 0.14\ N^2 \qquad (93)$$

The differences between these estimates are attributable to the reduction in the number of operations introduced by the overlap check. For example, the time in phase (2) is not dependent on K^2N^2 because the area covered by a finite element pair in the (r,s) integration domain is <u>inversely</u> proportional to N^2.

8. CONCLUSIONS

We have derived a variational principle for the pressure on the exterior surface of an arbitrary body in harmonic motion. The principle which was obtained from the Kirchhoff-Helmholtz integral theorem, relates the surface pressure directly to the normal component of velocity. It avoids the need to solve the wave equation for the exterior domain subject to compatibility conditions at the surface. Current approximate methods for decoupling the surface pressure from the exterior domain, such as the doubly asymptotic approximation, rely on assumptions regarding the relationship between pressure and particle velocity. In contrast, the variational principle relies on no assumptions beyond those associated with linearized acoustic theory.

We specialized the variational principle to the case of a thin circular disk vibrating transversely. Comparable techniques may be employed to describe other thin bodies, such as curved panels and bars. We developed two techniques for applying the variational principle to the evaluation of the surface pressure. Both involved expansion of the pressure in a modal series, with the difference being whether the modes are analytical functions covering the entire surface, or finite elements covering only a segment of the surface.

The results obtained by using analytical modes were found to be remarkably accurate. Trial functions that match the singularities encountered in diffraction of an edge give the best results, but reasonable convergence was obtained for more general functions. The finite element formulation predicted the pressure less accurately. We concluded from computations for a variety of parameters that piecewise linear finite elements generate significant numerical noise when the number of elements is increased. We expect that the usage of polynomial elements would significantly improve the results.

We employed both formulations to obtain the pressure for nondimensional radii in the range $ka \leq 5$. We could have presented results for $ka > 5$, but have not done so because of a lack of data to use for a comparison. The trends suggested by the present results are that higher ka would require more

analytical modes or elements in order to accommodate the more rapid spatial variation of pressure. Correspondingly, the mesh of integration points would have to be finer.

The variational principle may be extended to other geometrics. One of our current efforts is devoted to a finite length cylinder in axial vibration. Systems lacking axisymmetry, such as a transversely vibrating cylinder, may be treated by these techniques, except that more modes and more numerical integration points would be required to treat the higher dimensionality of such systems.

Another of our current efforts is devoted to coupling the variational principle to a modal description of structural vibration. The result will be a highly accurate, yet computationally simple, description of fluid-structure interaction. We have also shown that the variational principle may be readily modified to describe the scattering of acoustic waves.

BIBLIOGRAPHY

E. Ar and R.E. Kleinman, The exterior Neumann problem for the three-dimensional Helmholtz equation, Archiv Rat. Mech. Anal. 23, 218-236 (1966).

B. B. Baker and E. T. Copson, The Mathematical Theory of Huygens' Principle, 2nd Edition, Oxford, 1950, pp. 186-189, see footnote p. 186 re Levine and Schwinger 1948, 1949 Phys. Rev. papers. "They consider the case when the wavefunction is required to vanish on the screen, and use a rather different notation. The analysis in the former paper is rather obscure, as many of the integrals involved appear to diverge."

R. P. Banaugh and W. Goldsmith, Diffraction of steady acoustic waves by surfaces of arbitrary shape, J. Acoust. Soc. Am. 35(10) 1590-1601 (1963).

J. Bazer and A. Brown, Diffraction of scalar waves by a circular aperture, IRE Trans. on Antennas and Propagation AP-7, special supplement, 12-20 (1959).

J. Bazer and H. Hochstadt, Diffraction of scalar waves by a circular aperture. II, Commun. Pure Appl. Math. 15, 1-33 (1962).

W. A. Bell, W. L. Meyer, and B. T. Zinn, Predicting the acoustics of arbitrarily shaped bodies using an integral approach, AIAA Journal 15(6), 813-820, June 1977.

W. A. Bell, W. L. Meyer, B. T. Zinn, and M. P. Stallybrass, Boundary integral solutions of three-dimensional acoustic radiation problems, J. Sound Vib. 59, 245-262 (1978).

W. A. Bell, W. L. Meyer, B. T. Zinn, and M. P. Stallybrass, Prediction of the sound field radiated from axisymmetric surfaces, J. Acous. Soc. Am. 65, 631-638 (1979).

J. M. Blatt and J. D. Jackson, On the interpretation of neutron-proton scattering data by the Schwinger variational method, Phys. Rev. 76(1) 18-37 (1949).

C. J. Bouwkamp, Diffraction Theory, Reports on Progress in Physics (The Physical Society, London) 17, 35-100 (1954). [p.40: "This procedure eventually leads to all kinds of divergent integrals (Baker and Copson, 1950, p. 186) and should therefore be neglected." p. 62 "Fortunately, the divergent integrals that occur in Levine and Schwinger's papers are easy to eliminate (Bouwkamp 1953) without affecting the numerical results."]

C. J. Bouwkamp, Theoretical and numerical treatment of diffraction through a circular aperture, IEEE Trans. Antennas and Propagation AP-18(2), 152-176 (1970). [This is a translation of Bouwkamp's doctoral dissertation, University of Groningen, 1941.]

C. J. Bouwkamp, A note on singularities occurring at sharp edges in electromagnetic diffraction theory, Physica 12(7), 467-474 (1946).

C. J. Bouwkamp, Diffraction Theory. A Critique of Some Recent Developments, New York University, Mathematics Research Group Report No. EM-50. See esp. pp. 23-24 and note the reference to Baker and Copson, p. 184.

C. J. Bouwkamp, On integrals occurring in the theory of diffraction of electromagnetic waves by a circular disc, Nederl. Akad. Wetensch. Proc. 53, 654- (1950).

G. B. Brundrit, A solution to the problem of scalar scattering from a smooth, bounded obstacle using integral equations, Quart. Journ. Mech. and Applied Math. 18(4), 473-489 (1965).

A. J. Burton, Numerical solution of scalar diffraction problems, in Numerical Solution of Integral Equations, edited by L. M. Delves and J. Walsh, Clarendon Press, Oxford, 1974, Chapter 21, 275-290.

A. J. Burton, The solution of Helmholtz' equation in exterior domains using integral equations, National Physical Laboratory, Division of Numerical Analysis and Computing, Teddington, Middlesex, NPL Report NAC30, January 1973.

A. J. Burton and G. F. Miller, The application of integral equation methods to the numerical solution of some exterior boundary-value problems, Proc. Roy. Soc. Lond. A 323, 201-210 (1971).

N. Chako, On the evaluation of certain integrals and their application to diffraction theory, Acta Phys. Polon. 24, 611-620 (1963).

N. Chako, Diffraction of electromagnetic waves by circular apertures and discs. Integral representation method. I. Acta Phys. Polon. 24, 621-627 (1963).

L. H. Chen and D. G. Schweikert, Sound radiation from an arbitrary body, J. Acoust. Soc. Am. 35(10), 1626-1632 (October 1963).

G. Chertock, Sound radiation from vibrating surfaces, J. Acoust. Soc. Am. 36(7), 1305-1313 (July 1964).

G. Chertock, Integral equation methods in sound radiation and scattering from arbitrary surfaces, Report 3538, Department of the Navy, Naval Ship Research and Development Center, Washington, D.C. 20034, Report 3538, June 1971.

G. Chertock, Solutions for sound-radiation problems by integral equations at the critical wavenumbers, J. Acoust. Soc. Am. 47(1, pt. 2), 387-388 (January 1970).

G. Chertock, Convergence of iterative solutions to integral equations for sound radiation, Quart. Appl. Math. 26, 268-272 (1968).

G. Chertock and M. A. Grosso, Some numerical calculations of sound radiation from vibrating surfaces, David Taylor Model Basin Report 2109, March 1966.

R. V. Churchill, Complex Variables and Applications, New York: McGraw-Hill, 162-164 (1960).

W. D. Collins, On the solution of some axisymmetric boundary value problems by means of integral equations, V. Some scalar diffraction problems for circular discs. Quart. J. Mech. Appl. Math. 14, 101-117 (1961).

L. G. Copley, Integral Equation method for radiation from vibrating bodies, J. Acoust. Soc. Amer. 41(4), 807-816 (1967).

L. G. Copley, Fundamental results concerning integral representations in acoustic radiation, J. Acoust. Soc. Am. 44(1), 28- (1968).

L. G. Copley and H. A. Schenck, Vanishing of the surface-pressure contribution to the Helmholtz integral, J. Acoust. Soc. Am. 44(1), 288-289 (1968).

E. T. Copson, An integral equation method of solving the plane diffraction problem, Proc. Roy. Soc. A186, 100- (1946).

P. J. Davis and I. Polonsky, Numerical interpolation, differentiation, and integration, in M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover Publication, ninth printing, 1972, pp. 875-924.

A. Edelyi, Variational principles in the mathematical theory of diffraction, Atti. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 87, 281-293 (1953).

F. H. Fenlon, Calculation of the acoustic radiation field at the surface of a finite cylinder by the method of weighted residuals, Proc. IEEE 57(3). 291-306, March 1969.

P. J. T. Filippi, Layer potentials and acoustic diffraction, J. Sound Vib. 54(4), 473-500 (1977).

P. Filippi and G. Dumery, Theoretical and numerical study of diffraction by a thin screen, Acustica 21, 344-350 (1969).

T. L. Geers, Residual potential and approximate methods for three-dimensional fluid-structure interaction problems, J. Acoust. Soc. Am. 49(5), 1505-1510 (1971).

E. Gerjuoy, A. R. P. Rau, and L. Spruch, A unified formulation of the construction of variational principles, Rev. Mod. Phys. 55(3), 725-774 (July 1983).

J. Giroire and J. C. Nedelec, Numerical solution of an exterior Neumann problem using a double layer potential, Mathematics of Computation 32(144), 973-990 (October 1978).

G. M. L. Gladwell, A variational formulation of damped acousto-structural vibration problems, J. Sound Vib. 4, 172- (1966).

G. M. L. Gladwell and V. Mason, Variational finite element calculation of the acoustic response of a rectangular panel, J. Sound Vib. 14(1), 115-135 (1971).

G. L. M. Gladwell, On energy and complementary energy formulations of acoustic and structural vibration problems, J. Sound Vib. 3(3), 233-241 (1966).

D. Greenspan and P. Werner, A numerical method for the exterior Dirichlet problem for the reduced wave equation, Archiv for Rational Mechanics and Analysis 23, 288-316 (1966).

M. A. Hamdi, Une formulation variationelle par equations integrales pour la resolution de l'equation de Helmholtz avec des conditions aux limites mixtes, Compt. Rend. Acad. Sci. Paris 292(II), 17-20 (January 5, 1981).

P. C. Hammer, O. J. Marlowe, and A. H. Stroud, Numerical integration over simplexes and cones, Math. Tables Aids Comput. 10, 130-137 (1956).

S. Hanish, A review of world contributions from 1945 to 1965 to the theory of acoustic radiation - Chapter II. Three dimensional harmonic radiation, U. S. Naval Research Laboratory, AD621125, August 1965.

E. B. Hansen, Diffraction of a scalar wave by a plane circular disk, in Symposium on Electromagnetic Theory and Antennas, 39-43, Pergamon Press (1963).

H. S. Heaps, Diffraction of an acoustical wave obliquely incident upon a circular disk, J. Acoust. Soc. Am. 26(5), 707 (September 1954).

A. E. Heins and R. C. MacCamy, On the scattering of waves by a disk, Zeitschrift fur angewandte Mathematik und Physik 11, 249-264 (1960).

J. L. Hess, Calculation of acoustic fields about arbitrary three-dimensional bodies, McDonnell-Douglas Rep. No. DAC 66901 (March 1968)

A. T. de Hoop, On the scalar diffraction by a circular aperture in an infinite plance screen, Appl. Sci. Res. B. Electrophysics, Acoustics, Optics, Mathematical Methods B4, 151-160 (1955).

A. T. de Hoop, On integrals occurring in the variational formulation of diffraction problems, Koninklijke Nederlandse Akademie van Wetenschappen, Proceedings, Ser. B, Physical Sciences 58, 325-329 (1955).

J. T. Hunt, M. R. Knittel, and D. Barach, Finite element approach to acoustic radiation from elastic structures, J. Acoust. Soc. Am. 55(2), 269-280 (February 1974).

M. A. Jaswon, Integral equation methods in potential theory I, Proc. Roy. Soc. A275, 23-32 (1963).

D. S. Jones, A critique of the variational method in scattering problems, IRE Transactions on Antennas and Propagation, AP-4(3), 297-301 (1956).

D. S. Jones, A new method for calculating scattering with particular reference to the circular disc, Commun. Pure Appl. Math. 9, 713-746 (1956).

D. S. Jones, Diffraction at high frequencies by a circular disk, Proc. Camb. Phil. Soc. 61, 223-244 (1965).

D. S. Jones, Integral equations for the exterior acoustic problem, Qurt. J. Mech. Appl. Math. 27(1), 129-142 (1974).

J. B. Keller, Diffraction by an aperture. I. J. Appl. Phys. 28, 426-444 (1957); errata, 29, 744 (1958).

O. D. Kellogg, Foundations of Potential Theory, Dover, 160-164 (1953).

J. B. Keller, R. M. Lewis and B. D. Seckler, Diffraction by an aperture II, J. Appl. Phys. 28, 570-579 (1957).

L. V. King, On the acoustic radiation pressure on circular discs: inertia and diffraction corrections Proc. Roy. Soc. A153, 1-16 (1935).

R. E. Kleinman and G. F. Roach, Boundary integral equations for the three-dimensional Helmholtz equation, SIAM Review 16(2), 214-236 (1974).

R. D. Kodis, An introduction to variational methods in electromagnetic scattering, J. Soc. Indust. Appl. Math. 2(2), 89-112 (1954).

W. Kohn, Variational methods in nuclear collision problems, Phys. Rev. 74(11), 1763-1772 (1948).

M. Kotani, An acoustical problem relating to the theory of the Rayleigh disc, Proc. Phys. Math. Soc. (Japan) 15, 30-57 (1933).

R. Kussmaul, A numerical method for the solution of the Neumann exterior space problem for the Helmholtz oscillation equation,
Computing 4, 246-273 (1969).

H. Lass, Vector and Tensor Analysis, McGraw-Hill (1950)

R. Leis, On the Neumann boundary value problem for the Helmholtz oscillation equation, Archiv Rat. Mech. Anal. 2, 101-113 (1958).

A. Leitner, Diffraction of sound by a circular disk, J. Acoust. Soc. Amer. 21(4), 331-334 (July 1949).

H. Levine, Variational principles in acoustic diffraction theory, J. Acoust. Soc. Am. 22(1) 48-55 (January 1950).

H. Levine, Variational principles in potential theory, Appl. Sci. Res. Sec. A,15, 371-391 (1965?)

H. Levine and C. H. Papas, Theory of the circular diffraction antenna, J. Appl. Phys. 22(1), 29-43 (1951).

H. Levine and J. Schwinger, On the theory of electromagnetic wave diffraction by an aperture in an infinite plane conducting screen, Commun. Pure Appl. Math. 3(4), 355-391 (December 1950).

H. Levine and J. Schwinger, On the theory of diffraction by an aperture in an infinite plane screen, Part I, Physical Review 74(8), 958-974 (October 15, 1948); Part II, 75(9), 1423-1432 (May 1, 1949).

H. Levine and J. Schwinger, On the transmission coefficient of a circular aperture, Phys. Rev., 1608-1609 (1949).

A. Levitas and M. Lax, Scattering and absorption by an acoustic strip, J. Acoust. Soc. Am. 23(3), 316-322 (May 1951).

B. A. Lippmann and J. Schwinger, Variational principles for scattering processes I, Phys. Rev. 79(1), 469-480 (1950).

J. N. Lyness, An error functional expansion for N-dimensional quadrature with an integrand function singular at a point, Math. Comp. 30, 1-23 (1976).

J. N. Lyness, Applications of extrapolation techniques to multidimensional quadrature of some integrand functions with a singularity, J. Computational Physics 20, 346-364 (1976).

J. M. McCormick and M. L. Baron, Sound radiation from submerged cylindrical shells of finite length, J. Engineering for Industry, Trans. ASME (Series B), Vol. 87, 393-405 (1965).

A.-W. Maue, On the formulation of a general diffraction problem by an integral equation, Zeit. Phys. 126, 601-618 (1949).

K. K. Mei and J. van Bladel, Scattering by perfectly conducting rectangular cylinders, IEEE Trans. Antennas and Propagation AP-11, 185-192 (1963).

J. Meixner, The edge condition in the theory of diffraction of electromagnetic waves at perfectly conducting screens, Ann. Phys. (6)6, 1-7 (1949).

J. Meixner, Rigorous theory of dirrfaction of electromagnetic waves at a perfectly conducting circular orifice, Zeit. Naturforschung, 3A, 506- (1948).

W. L. Meyer, W. A. Bell, M. P. Stallybrass, and B. T. Zinn, Prediction of the sound field radiated from axisymmetric surfaces, J. Acoust. Soc. Am. 65(3), 631-638 (March 1979).

W. L. Meyer, W. A. Bell, B. T. Zinn, and M. P. Stallybrass, Boundary integral solutions of three dimensional acoustic radiation problems, J. Sound and Vibration, 59(2), 245-262 (1978).

J. W. Miles, on the diffraction of an electromagnetic wave through a plane screen, J. Appl. Phys. 20(8), 760-771 (1949). Errata: On the diffraction of an electromagnetic wave through a plane screen, 21(5) 468 (1950)

J. W. Miles, On acoustic diffraction cross sections for oblique incidence, J. Acoust. Soc. Am. 24(3), 324-324 (May 1952).

J. W. Miles, On acoustic diffraction through an aperture in a plane screen, Acustica 2, 287-291 (1952).

L. M. Milne-Thomson, Elliptic Integrals, in M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover Publications, ninth printing, 1972, pp. 587-626.

K. M. Mitzner, Acoustic scattering from an interface between media of greatly different density, J. Math. Phys. 7, 2053-2060 (1966).

K. M. Mitzner, Numerical solution for transient scattering from a hard surface of arbitrary shape - retarded potential shape, J. Acoust. Soc. Am. 42, 391-397 (February 1967).

P. M. Morse and H. Feshbach, Methods of Theoretical Physics, Vol. 2, McGraw-Hill, 1953, pp. 1136-1137.

P. M. Morse, Transmission of sound through a circular membrane in a plane wall, J. Acoust. Soc. Am. 40(2), 354-366 (1966).

J. C. Nedelec and J. Planchard, Une methode variationnelle d'elements finis pour la resolution numerique d'un probleme exterieur dans R3, Rev. Francaise Informat. Recherche Operationnelle Ser. Rouge 3, v. 7;, 105-129 (1973).

T. Nimura and Y. Watanabe, Effect of a finite circular baffle board on acoustic radiation, J. Acoust. Soc. Am. 25(1), 76-80 (January 1953).

O. I. Panich, On the question of the solvability of the exterior boundary value problem for the wave equation and Maxwell's equations, Uspekhi Mat. Nauk. 20(1), 221-226 (1965).

A. D. Pierce, Acoustics: an introduction to its physical principles and applications, McGraw-Hill Book Co., New York, 1981.

P. H. Rogers, Formal solution of the surface Helmholtz integral equation at a nondegenerate characteristic frequency, J. Acoust. Soc. Am. 54(6), 1662-1666 (1973).

P. H. Rogers, Formal solution of the surface Helmholtz integral equation at a nondegenerate characteristic frequency, J. Acoust. Soc. Am. 54, 1662-1666 (1973).

H. A. Schenck, Improved integral formulation for acoustic radiation, J. Acoust. Soc. Am. 44(1), 41-58 (1968).

R. Seznec, Diffraction of sound around barriers: use of the boundary elements technique, Journal of Sound and Vibration, 73(2), 195-209 (1980).

F. B. Sleator, A variational solution to the problem of scalar scattering by a prolate spheroid, J. Math. Phys. 39, 105-120 (1960).

A. Sommerfeld, The freely vibrating piston-membrane, Ann. Phys. (5)42, 389-420 (1942/43).

R. D. Spence, The diffraction of sound by circular disks and apertures, J. Acoust. Soc. Am. 20(4), 380-386 (July 1948).

M. P. Stallybrass, On a pointwise variational principle for the approximate solution of linear boundary value problems, J. of Math. and Mech. 16(11), 1247-1286 (1967).

A. Storruste and H. Wergeland, On two complementary diffraction problems, Phys. Rev. 73, 1397-1398 (1948).

G. T. Symm, Integral equation methods in potential theory II, Proc. Roy . Soc. A275, 33-46 (1963).

C. T. Tai, Electromagnetic back-scattering from cylindrical wires, J. Appl. Phys. 23(8) 909-916 (August 1952)

F. Ursell, On the exterior problems of acoustics, Proc. Camb. Phil. Soc. 74, 117-125 (1973).

F. M. Wiener, The diffraction of sound by rigid disks and rigid square plates, J. Acoust. Soc. Am. 21(4), 334-347 (July 1949).

W. Williams, N. G. Parke, D. A. Moran, and C. H. Sherman, Acoustic radiation from a finite cylinder, J. Acoust. Soc. Am. 36, 2316-2322 (December 1964).